

# ABSTRACT STOCHASTIC EVOLUTION EQUATIONS IN M-TYPE 2 BANACH SPACES

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This paper devotes to studying abstract stochastic evolution equations in M-type 2 Banach spaces. First, we handle nonlinear evolution equations with multiplicative noise. The existence and uniqueness of local and global mild solutions under linear growth and Lipschitz conditions on coefficients are presented. The regular dependence of solutions on initial data is also studied. Second, we investigate linear evolution equations with additive noise. The existence and uniqueness of strict and mild solutions and their regularity are shown. Finally, we explore semilinear evolution equations with additive noise. We concentrate on the existence, uniqueness and regular dependence of solutions on initial data.

**1. Introduction.** The theory of stochastic partial differential equations in Hilbert spaces has been studied from 1970s. The basic theoretical problem on existence and uniqueness of solutions and the problem on regularity and regular dependence on initial data of solutions are still of great interest today. Two main approaches are known to the abstract stochastic evolution equations, namely, the variational methods and the semigroup methods. Some early work in the first approach are due to Bensoussan and Temam [2, 3]. The fundamental work on monotone stochastic evolution equations is also due to Pardoux [21, 22]. There are many other important contributions in this approach, for examples Krylov-Rosovskii [19], Prévôt and Röckner [25], Viot [28, 29], Gawarecki and Mandrekar [16] and references therein.

Let us introduce the second approach. The semigroup methods, which were initiated by the invention of the analytic semigroups in the middle of the last century, are characterized by precise formulas representing the solutions of the Cauchy problem for deterministic evolution equations (see Hille [17] and Yosida [31]). The analytical semigroup  $S(t) = e^{-tA}$  generated by a linear operator  $-A$  provides directly a fundamental solution to the

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<sup>\*</sup>This work is supported by the Japan Society for the Promotion of Science.

<sup>†</sup>This work is supported by Grant-in-Aid for Scientific Research (No. 20340035) of the Japan Society for the Promotion of Science.

*MSC 2010 subject classifications:* Primary 60H15, 35R60; secondary 47D06

*Keywords and phrases:* stochastic evolution equations, analytic semigroups, M-type 2 Banach spaces

Cauchy problem for an autonomous linear evolution equation

$$\begin{cases} \frac{dX}{dt} + AX = F(t), & 0 < t \leq T, \\ X(0) = X_0, \end{cases}$$

and the solution is given by the formula  $X(t) = S(t)X_0 + \int_0^t S(t-s)F(s)ds$ . Similarly, a solution to the Cauchy problem for an autonomous nonlinear evolution equation

$$\begin{cases} \frac{dX}{dt} + AX = F(t, X), & 0 < t \leq T, \\ X(0) = X_0, \end{cases}$$

can be obtained as a solution of an integral equation  $X(t) = S(t)X_0 + \int_0^t S(t-s)F(s, X(s))ds$ . For these problems, the solution formulas provide us important information on solutions such as uniqueness, regularity, smoothing effect and so forth. Especially, for nonlinear problems, one can derive Lipschitz continuity of solutions with respect to the initial values, even their Fréchet differentiability. This powerful approach has been used in the study of stochastic evolution equations in Hilbert spaces. Some early work was proposed by Dawson [10, 11], Curtain and Falb [5]. More recent important contributions are due to Da Prato and his collaborations, see for examples [6, 7, 8, 9], and references therein.

In this paper, we study abstract stochastic evolution equations in M-type 2 Banach spaces by using the semigroup methods. Let  $E$  be an M-type 2 real separable Banach space and  $\mathcal{B}(E)$  be the Borel  $\sigma$ -field on  $E$ . Let  $\{w_t, t \geq 0\}$  be a one-dimensional Brownian motion on a complete probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  with a filtration  $\{\mathcal{F}_t\}_{t \geq 0}$  satisfying the usual conditions (see for example Arnold [1], Friedman [15], Karatzas-Shreve [18]). We proceed to study abstract stochastic evolution equations of the form

$$(1.1) \quad \begin{cases} dX + AXdt = F(t, X)dt + G(t, X)dw_t, \\ X(0) = \xi, \end{cases}$$

where  $A: \mathcal{D}(A) \subset E \rightarrow E$  is a densely defined, closed linear operator in  $E$ . The functions  $F$  and  $G$  are  $E$ -valued random variables. The initial value  $\xi$  is an  $\mathcal{F}_0$ -measurable random variable.

A linear form of (1.1) in an M-type 2 and UMD Banach space, i.e.  $F(t, x) = F(t)$  is depending only on  $t$  and  $G(t, x) = Bx$ , where  $B$  is a linear operator from the space to itself, is investigated in Brzeźniak [4]. The author showed a sufficient condition on the operator  $A$  and the linear operator  $B$ , which is assumed to be bounded as an operator from  $\mathcal{D}(A)$  into  $\mathcal{D}_A(\frac{1}{2}, 2)$ , under which (1.1) has a strict solution.

Our objective is to study the existence and uniqueness, regularity and dependence on initial data of solutions of (1.1). Our contribution is threefold. First, we handle nonlinear evolution equations with multiplicative noise. Second, we investigate linear evolution equations with additive noise. Finally, we concentrate on semilinear evolution equations with additive noise. The work on linear and semilinear evolution equations with multiplicative noise is in preparation [26, 27].

Let us describe the content of the present paper. In Section 2, we introduce the stochastic integral in the M-type 2 Banach space  $E$ , concepts of solutions of (1.1), function spaces with values in  $E$  and an integral inequality of Volterra type.

In Section 3, we show existence of solutions of (1.1) as well as regular dependence of solutions on initial data under some conditions on the coefficients  $F$  and  $G$ . This section contains three subsections. In the first subsection, we assume that  $F$  and  $G$  satisfy the linear growth and Lipschitz conditions. Theorem 3.1 gives the existence of global mild solutions of (1.1). The proof of the theorem is similar to that in Da Prato-Zabczyk [9]. In the second subsection, we assume that  $F$  and  $G$  satisfy the local linear growth and local Lipschitz conditions. Theorem 3.4 shows the existence of maximal local mild solutions of (1.1). The proof is based on Lemma 3.2, Lemma 3.3 and Theorem 3.1. Noting that these lemmas come from earlier results in the theory of ordinary differential equations (see for example Friedman [15]). Theorem 3.5 gives the dependence of the maximal local mild solution on the initial data. In the last subsection, we assume that  $F$  and  $G$  satisfy the linear growth and local Lipschitz conditions. Theorem 3.6 demonstrates the existence of global mild solutions of (1.1). Theorem 3.7 presents the dependence of the global mild solutions on the initial data.

In Section 4, we concentrate on a class of equations of (1.1), namely, linear evolution equations with additive noise. We assume that  $F(t, x) = F(t)$  and  $G(t, x) = G(t)$  are depending only on  $t$ . These functions are considered in the spaces  $\mathcal{F}^{\beta, \sigma}((0, T]; E)$  and  $\mathcal{B}((0, T]; E)$ , which will be defined in Section 2. Theorem 4.2 gives the existence of strict solutions. Theorem 4.4 explores the regularity of mild solutions without the assumption  $|AS(t)| \leq c_\delta t^{-\delta}$ ,  $t \in [0, T]$ ,  $\delta \in (0, \beta)$  of Theorem 4.2.

In Section 5, we set  $F(t, x) = F_1(x) + F_2(t)$  and  $G(t, x) = G(t)$ , where  $F_1, F_2$  and  $G$  are depending only on  $x$  and  $t$ , respectively. The corresponding equation (5.1) is then the form of semilinear evolution equations with additive noise. We suppose that the function  $F_1$  is defined only on a subset of the space  $E$ , namely  $\mathcal{D}(F_1) = \mathcal{D}(A^\eta)$ , and that  $F_1$  satisfies a Lipschitz condition on its domain (see (5.2)). To treat (5.1) we require that the initial

condition takes values in a smaller space, say  $\mathcal{D}(A^\beta)$ . Theorem 5.1 proves the existence of mild solutions in the function space  $\mathcal{C}((0, T_{F,G,\xi}]; \mathcal{D}(A^\eta)) \cap \mathcal{C}([0, T_{F,G,\xi}]; \mathcal{D}(A^\beta))$ . Theorem 5.3 gives a more stronger regularity under more regular initial values. Theorem 5.4 presents some results on regular dependence of solutions on initial data. Theorem 5.5 shows the existence of solutions for a critical case of the Lipschitz condition on  $F_1$ . Finally, Theorem 5.8 explores the regular dependence of solutions on initial data for this critical case.

## 2. Preliminary.

2.1. *Stochastic integrals in  $M$ -type 2 Banach spaces.* This subsection reviews the construction and some properties of the stochastic integral in  $M$ -type 2 real separable space  $E$ . All details in this subsection one can find in the work of Dettweiler [12, 13, 14], Pisier [23, 24] and Brzeźniak [4].

DEFINITION 2.1 (Pisier [24]). A Banach sapce  $E$  is of  $M$ -type 2 (or martingale type 2) if there is a constant  $c(E)$  such that for all  $E$ -valued martingale  $\{M_n\}_n$  the inequality

$$\sup_n \mathbb{E}|M_n|^2 \leq c(E) \sum_{n \geq 0} \mathbb{E}|M_n - M_{n-1}|^2$$

holds with the convention  $M_{-1} = 0$ .

EXAMPLE 2.2. Every  $L^p$  space with  $p \in [2, \infty)$  is of  $M$ -type 2.

Let us first define the stochastic integral for step functions. Let  $f : [0, T] \times \Omega \rightarrow E$  be an adapted random step function, i.e. there exist sequences  $\{t_i\}_0^n : 0 = t_0 < \dots < t_n = T$  and  $\{f_i\}_0^{n-1} : f_i \in L^2(\Omega, \mathcal{F}_{t_i}, \mathbb{P}; E)$  such that  $f(s) = f_i$  a.e. for  $s \in [t_i, t_{i+1})$ . Then the stochastic integral of  $f$  on  $[0, T]$  with respect to  $w_t$  is defined by

$$I_T(f) := \int_0^T f(t)dw_t = \sum_0^{n-1} (w_{t_{i+1}} - w_{t_i})f_i.$$

It is obvious that  $I_T(f)$  is  $\mathcal{F}_T$ -measurable. In addition, by putting

$$M_k = \sum_{i=0}^k (w_{t_{i+1}} - w_{t_i})f_i,$$

then  $I_T(f) = M_{n-1}$  and  $\{M_k\}_{k=0}^{n-1}$  is a martingale. In view of Definition 2.1, we have

$$\begin{aligned}
 (2.1) \quad \mathbb{E}|I_T(f)|^2 &\leq \sup_k \mathbb{E}|M_k|^2 \leq c(E) \sum_{k=0}^{n-1} \mathbb{E}|(w_{t_{k+1}} - w_{t_k})f_i|^2 \\
 &= c(E) \sum_{k=0}^{n-1} \mathbb{E}|w_{t_{k+1}} - w_{t_k}|^2 \mathbb{E}|f_i|^2 \\
 &= c(E) \sum_{k=0}^{n-1} (t_{k+1} - t_k) \mathbb{E}|f_i|^2 \\
 &= c(E) \int_0^T \mathbb{E}|f(t)|^2 dt.
 \end{aligned}$$

Denote by  $\mathcal{P}_\infty$  the predictable  $\sigma$ -field on  $\Omega_\infty = [0, \infty) \times \Omega$  generated by sets of the form

$$(s, t] \times K_1, \quad 0 \leq s < t < \infty, K_1 \in \mathcal{F}_s \text{ and } \{0\} \times K_2, \quad K_2 \in \mathcal{F}_0,$$

and denote by  $\mathcal{P}_T$  the restriction of  $\mathcal{P}_\infty$  to  $\Omega_T = [0, T] \times \Omega$ . A process  $\phi$  is called predictable if it is measurable from  $(\Omega_T, \mathcal{P}_T)$  into  $(E, \mathcal{B}(E))$ . We then denote by  $\mathcal{N}^2(0, T)$  the set of all  $E$ -valued predictable processes  $\phi$  such that  $\mathbb{E} \int_0^T |\phi(t)|^2 dt < \infty$ . Thanks to the inequality (2.1), one can define the stochastic integral for functions in  $\mathcal{N}^2(0, T)$  (a set  $\mathcal{N}^2(a, b)$ ,  $a, b \in \mathbb{R}$ , and the integral for functions in  $\mathcal{N}^2(a, b)$  are defined similarly). Indeed, due to (2.1), it is not hard to show that the limit in the following definition exists and does not depend on the actual choice of step functions.

**DEFINITION 2.3** (stochastic integrals). Let  $f \in \mathcal{N}^2(0, T)$ . The stochastic integral of  $f$  on  $[0, t]$ ,  $0 \leq t \leq T$  is defined by

$$I_t(f) := \int_0^t f(s) dw_s = \lim_{n \rightarrow \infty} \int_0^t \phi_n(s) dw_s \quad (\text{limit in } L^2(\mathbb{P}))$$

where  $\{\phi_n\}_n$  is a sequence of step functions such that

$$\lim_{n \rightarrow \infty} \mathbb{E} \int_0^t |f(s) - \phi_n(s)|^2 ds = 0.$$

**PROPOSITION 2.4.** Let  $f \in \mathcal{N}^2(0, \infty)$ . Then

- (i)  $\mathbb{E}|I_t(f)|^2 \leq c(E) \int_0^t \mathbb{E}|f(s)|^2 ds, \quad t \geq 0.$
- (ii)  $\{I(f)\}_t$  is an  $E$ -valued continuous martingale and  $I(f) \in \mathcal{N}^2(0, \infty)$ .

(iii) For any  $p > 1, T > 0$

$$\mathbb{E} \sup_{[0, T]} \left| \int_0^t \phi(s) dw_s \right|^p \leq \left( \frac{p}{p-1} \right)^p c_p(E) \mathbb{E} \left[ \int_0^T |\phi(s)|^2 ds \right]^{\frac{p}{2}},$$

where  $c_p(E)$  is some constant depending only on  $p, E$ .

**2.2. Concept of solutions.** Throughout this paper, if not specified, we always assume that  $F$  and  $B$  are measurable from  $(\Omega_T \times E, \mathcal{P}_T \times \mathcal{B}(E))$  into  $(E, \mathcal{B}(E))$ . In addition, we assume that  $(-A)$  generates a strongly continuous semigroup  $S(t) = e^{-tA}, t \geq 0$  on  $E$ . We then set  $M_t = \sup_{0 \leq s \leq t} |S(s)|$ .

Following the definition of local solutions to stochastic differential equations (e.g. Arnold [1], Friedman [15], Mao [20]), we present a definition of local mild solutions for (1.1).

**DEFINITION 2.5.** Let  $\tau$  be a stopping time such that  $\tau \leq T$  a.s. A predictable  $E$ -valued continuous process  $\{X(t), t \in [0, \tau]\}$  is called a local mild solution of (1.1) if the followings are satisfied.

- (i) There exists a sequence  $\{\tau_k\}_{k=1}^\infty$  of stopping times such that  $0 \leq \tau_k \leq \tau_{k+1}, k \geq 1$  and  $\lim_{k \rightarrow \infty} \tau_k = \tau$  a.s.
- (ii) For  $t \in [0, T]$  and  $k \geq 1$ ,  $S(t - \cdot)G(\cdot, X(\cdot)) \in \mathcal{N}^2(0, t \wedge \tau_k)$ , i.e.

$$\mathbb{E} \int_0^{t \wedge \tau_k} |S(t-s)G(s, X(s))|^2 ds < \infty.$$

- (iii) For  $t \in [0, T]$  and  $k \geq 1$

$$\int_0^{t \wedge \tau_k} |S(t-s)F(s, X(s))| ds < \infty \quad \text{a.s.}$$

- (iv) For  $t \in [0, T]$  and  $k \geq 1$

$$\begin{aligned} X(t \wedge \tau_k) = & S(t \wedge \tau_k) \xi + \int_0^{t \wedge \tau_k} S(t-s)F(s, X(s)) ds \\ & + \int_0^{t \wedge \tau_k} S(t-s)G(s, X(s)) dw_s \quad \text{a.s.} \end{aligned}$$

If in addition  $\lim_{t \uparrow \tau} |X(t)| = \infty$  on  $\{\tau < T\}$ , then  $\{X(t), t \in [0, \tau]\}$  is called a maximal local mild solution. A maximal local mild solution  $\{X(t), 0 \leq t < \tau\}$  is said to be unique if any other maximal local mild solution  $\{\bar{X}(t), 0 \leq t < \bar{\tau}\}$  is indistinguishable from it, means that  $\mathbb{P}\{\tau = \bar{\tau}\} = \mathbb{P}\{X(t) = \bar{X}(t) \text{ for every } t \in [0, \tau]\} = 1$ .

The following definition of global mild solutions is presented in Da Prato-Zabczyk [9] and Brzeźniak [4].

DEFINITION 2.6. A predictable  $E$ -valued continuous process  $X(t), t \in [0, T]$  is called a global mild solution (briefly, a mild solution) of (1.1) if  $X \in \mathcal{N}(0, T)$  and for every  $t \in [0, T]$

$$(2.2) \quad \begin{aligned} X(t) = & S(t)\xi + \int_0^t S(t-s)F(s, X(s))ds \\ & + \int_0^t S(t-s)G(s, X(s))dw_s \quad \text{a.s.} \end{aligned}$$

It is obvious that a maximal local mild solution  $\{X(t), t \in [0, \tau)\}$  of (1.1) is a mild solution on  $[0, T]$  if  $\tau = T$  a.s. and  $X$  satisfies (2.2) and is continuous at  $t = T$ .

DEFINITION 2.7. A predictable  $E$ -valued continuous process  $X(t), t \in [0, T]$  is called a strict solution of (1.1) if

- (i)  $G(\cdot, X(\cdot)) \in \mathcal{N}^2(0, T)$ ,
- (ii)  $|\int_0^t F(s, X(s))ds| < \infty, \quad t \in [0, T]$ ,
- (iii)  $X(t) \in D(A)$  and  $\int_0^t \mathbb{E}|X(s)|_{D(A)}^2 ds < \infty, \quad t \in (0, T]$ ,
- (iv) for every  $t \in (0, T]$

$$X(t) = \xi - \int_0^t AX(s)ds + \int_0^t F(s, X(s))ds + \int_0^t G(s, X(s))dw_s \quad \text{a.s.}$$

A strict (mild) solution  $\{X(t), 0 \leq t \leq T\}$  is said to be unique if any other strict (mild) solution  $\{\bar{X}(t), 0 \leq t \leq T\}$  is indistinguishable from it, means that  $\mathbb{P}\{X(t) = \bar{X}(t) \text{ for every } t \in [0, T]\} = 1$ .

2.3. *Function spaces with values in a Banach space.* Let  $I$  be an interval of the real line. By  $\mathcal{B}(I; E)$ , we denote the space of uniformly bounded  $E$ -valued functions on  $I$ . The space is a Banach space with the supremum norm

$$|G|_{\mathcal{B}(I; E)} = \sup_{t \in I} |G(t)|, \quad G \in \mathcal{B}(I; E).$$

Denotes by  $\mathcal{C}(I; E)$  the space of  $E$ -valued continuous functions. The following well-known result is used very often.

THEOREM 2.8. Let  $A$  be a closed linear operator of  $E$  and  $a, b \in \mathbb{R}, a \leq b$ .

(i) If  $G \in \mathcal{C}([a, b]; E)$  and  $AG \in \mathcal{C}([a, b]; E)$  then

$$A \int_a^b G(t) dt = \int_a^b AG(t) dt.$$

(ii) If  $G \in \mathcal{N}^2(a, b)$  and  $AG \in \mathcal{N}^2(a, b)$  then

$$A \int_a^b G(t) dw_t = \int_a^b AG(t) dw_t.$$

For an exponent  $\sigma > 0$ ,  $\mathcal{C}^\sigma([a, b]; E)$ ,  $a \leq b$  denotes the space of functions which are Hölder continuous on  $[a, b]$  with exponent  $\sigma$ . The space is equipped with norm

$$|G|_{\mathcal{C}^\sigma([a, b]; E)} = \sup_{a \leq s < t \leq b} \frac{|G(t) - G(s)|}{(t - s)^\sigma}.$$

The Kolmogorov test gives a sufficient condition for a stochastic process to be Hölder continuous.

**THEOREM 2.9** (Kolmogorov test, see e.g. Da Prato-Zabczyk [9]). *Let  $\zeta(t)$ ,  $t \in [0, T]$  be an  $E$ -valued stochastic process such that for some constants  $c > 0$ ,  $\epsilon_i > 0$ ,  $i = 1, 2$  and all  $t, s \in [0, T]$*

$$(2.3) \quad \mathbb{E}|\zeta(t) - \zeta(s)|^{\epsilon_1} \leq c|t - s|^{1+\epsilon_2}.$$

*Then  $\zeta$  has an version whose  $\mathbb{P}$ -almost all trajectories are Hölder continuous functions with an arbitrary exponent smaller than  $\frac{\epsilon_2}{\epsilon_1}$ .*

When the process  $\zeta(t)$  in Theorem 2.9 is a Gaussian process, one can weaken the condition (2.3).

**THEOREM 2.10.** *Let  $\zeta(t)$ ,  $t \in [0, T]$  be an  $E$ -valued Gaussian process such that  $\mathbb{E}\zeta(t) = 0$ ,  $t \geq 0$ , and that for some constants  $c > 0$ ,  $\epsilon \in (0, 1]$  and all  $t, s \in [0, T]$*

$$\mathbb{E}|\zeta(t) - \zeta(s)|^2 \leq c|t - s|^\epsilon.$$

*Then there exists a modification of  $\zeta$  with  $\mathbb{P}$ -almost all trajectories being Hölder continuous functions with an arbitrary exponent smaller than  $\frac{\epsilon}{2}$ .*

For two exponents  $0 < \sigma < \beta \leq 1$  we define a function space  $\mathcal{F}^{\beta, \sigma}((0, T]; E)$  as follows, see Yagi [30] ( $\mathcal{F}^{\beta, \sigma}((a, b]; E)$ ,  $a < b$  is defined similarly). The space  $\mathcal{F}^{\beta, \sigma}((0, T]; E)$  consists of all continuous function  $f(t)$  on  $(0, T]$  (resp.  $[0, T]$ ) when  $0 < \beta < 1$  (resp.  $\beta = 1$ ) with the following three properties:



1. When  $\beta < 1$ ,  $t^{1-\beta}f(t)$  has a limit as  $t \rightarrow 0$ .
2. The function  $f$  is Hölder continuous with exponent  $\sigma$  and with the weight  $s^{1-\beta+\sigma}$ , i.e.

$$\sup_{0 \leq s < t \leq T} \frac{s^{1-\beta+\sigma}|f(t) - f(s)|}{(t-s)^\sigma} = \sup_{0 \leq t \leq T} \sup_{0 \leq s < t} \frac{s^{1-\beta+\sigma}|f(t) - f(s)|}{(t-s)^\sigma} < \infty.$$

3.

$$(2.4) \quad \lim_{t \rightarrow 0} \sup_{0 \leq s \leq t} \frac{s^{1-\beta+\sigma}|f(t) - f(s)|}{(t-s)^\sigma} = 0.$$

Then  $\mathcal{F}^{\beta,\sigma}((0, T]; E)$  becomes a Banach space with norm

$$|f|_{\mathcal{F}^{\beta,\sigma}} = \sup_{0 \leq t \leq T} t^{1-\beta}|f(t)| + \sup_{0 \leq s < t \leq T} \frac{s^{1-\beta+\sigma}|f(t) - f(s)|}{(t-s)^\sigma}.$$

The following useful inequality follows the definition directly. For every  $f \in \mathcal{F}^{\beta,\sigma}((0, T]; E)$ ,  $0 < s < t \leq T$  we have

$$(2.5) \quad \begin{cases} |f(t)| \leq |f|_{\mathcal{F}^{\beta,\sigma}} t^{\beta-1}, \\ |f(t) - f(s)| \leq |f|_{\mathcal{F}^{\beta,\sigma}} (t-s)^\sigma s^{\beta-\sigma-1}. \end{cases}$$

In addition, it is not hard to show that

$$(2.6) \quad \mathcal{F}^{\gamma,\sigma}((0, T]; E) \subset \mathcal{F}^{\beta,\sigma}((0, T]; E), \quad 0 < \sigma < \beta < \gamma \leq 1.$$

The space  $\mathcal{F}^{\beta,\sigma}((0, T]; E)$  is not a trivial space. Indeed, we have

REMARK 2.11 (Yagi [30]). When  $0 < \sigma < \beta < 1$ ,  $f(t) = t^{\beta-1}g(t) \in \mathcal{F}^{\beta,\sigma}((0, T]; E)$ , where  $g(t)$  is any  $E$ -valued function on  $[0, T]$  such that  $g \in \mathcal{C}^\sigma([0, T]; E)$  and  $g(0) = 0$ . When  $0 < \sigma < \beta = 1$ , the space  $\mathcal{F}^{1,\sigma}((0, T]; E)$  includes the space of Hölder continuous functions with exponent  $\sigma$ .

2.4. *Integral inequality of Volterra type.* Let us introduce an useful inequality of Volterra type that will be used in this paper. The proof of the inequality can be found, for example, in Yagi [30].

LEMMA 2.12. *Let  $a \geq 0, b > 0, \mu_1 > 0$  and  $\mu_2 > 0$  be constants.*

(i) *Let  $\Gamma$  be the gamma function. Then the function defined by the series*

$$E_{\mu,\nu}(t) = \sum_{n=0}^{\infty} \frac{t^{n\nu}}{\Gamma(\mu_1 + n\mu_2)}, \quad 0 \leq t < \infty$$

satisfies an estimate

$$E_{\mu,\nu}(t) \leq \frac{2}{\Gamma_0 \nu_2} (1+t)^{2-\mu_1} e^{t+1}, \quad 0 \leq t < \infty,$$

where  $\Gamma_0 = \min_{0 < s < \infty} \Gamma(s)$ .

- (ii) Let  $\varphi(t, s)$  be a nonnegative continuous function defined for  $0 \leq s < t \leq T$ . If  $\varphi(t, s)$  satisfies the integral inequality

$$\varphi(t, s) \leq a(t-s)^{\mu_1-1} + b \int_s^t (t-r)^{\mu_2-1} \varphi(r, s) dr, \quad 0 \leq s < t \leq T,$$

then

$$\varphi(t, s) \leq a\Gamma(\mu_1)(t-s)^{\mu_1-1} E_{\mu_1, \mu_2}([b\Gamma(\mu_2)]^{\frac{1}{\mu_2}}(t-s)), \quad 0 \leq s < t \leq T.$$

### 3. Nonlinear evolution equations with multiplicative noise.

3.1. *Existence of global mild solutions under linear growth and Lipschitz conditions.* In this subsection, we shall show existence and uniqueness of global mild solutions of (1.1) under linear growth and Lipschitz conditions on  $F$  and  $G$ . The proof is similar to that in Da Prato-Zabczyk [9].

**THEOREM 3.1** (global existence). *Assume that  $F$  and  $G$  satisfy two conditions:*

- (i) *The linear growth condition*

$$(3.1) \quad |F(t, x)| + |G(t, x)| \leq c_1(1 + |x|), \quad x \in E, t \in [0, T].$$

- (ii) *The Lipschitz condition*

$$(3.2) \quad |F(t, x) - F(t, y)| + |G(t, x) - G(t, y)| \leq c_2|x - y|, \quad x, y \in E, t \in [0, T],$$

where  $c_i > 0$  ( $i = 1, 2$ ) are some positive constants. Suppose further that  $\mathbb{E}|\xi|^p < \infty$  for some  $p \geq 2$ . Then there exists a unique mild solution  $X(t)$  to (1.1) on  $[0, T]$ . Furthermore, it satisfies the estimate

$$(3.3) \quad \sup_{0 \leq t \leq T} \mathbb{E}|X(t)|^p \leq \alpha(1 + \mathbb{E}|\xi|^p),$$

where  $\alpha = \alpha(c_1, p, M_T, T) > 0$  is some constant depending only on  $c_1, p, M_T$  and  $T$ .

PROOF. We shall use the fixed point theorem and the Gronwall lemma. The proof is divided into three steps.

**Step 1.** Let us show existence of a mild solution. Put

$$(3.4) \quad \mathcal{Q}_1(Y)(t) = \int_0^t S(t-s)F(s, Y(s))ds,$$

$$(3.5) \quad \mathcal{Q}_2(Y)(t) = \int_0^t S(t-s)G(s, Y(s))dw_s,$$

$$\mathcal{Q}(Y)(t) = S(t)\xi + \mathcal{Q}_1(Y)(t) + \mathcal{Q}_2(Y)(t)$$

and let  $\mathcal{E}_p(0, \bar{T})(\bar{T} \leq T)$  be the set of all  $E$ -valued predictable process  $Y(t)$  on  $[0, \bar{T}]$  such that  $\sup_{t \in [0, \bar{T}]} \mathbb{E}|Y(t)|^p < \infty$ . Then up to indistinguishability,  $\mathcal{E}_p(0, \bar{T})$  is a Banach space with norm

$$\|Y\|_{p, \bar{T}} = \left[ \sup_{t \in [0, \bar{T}]} \mathbb{E}|Y(t)|^p \right]^{\frac{1}{p}}.$$

Let us show that  $\mathcal{Q}(\mathcal{E}_p(0, \bar{T})) \subset \mathcal{E}_p(0, \bar{T})$ . Indeed, by using Hölder inequality, we have

$$\begin{aligned} \|\mathcal{Q}_1(Y)\|_{p, \bar{T}}^p &\leq \sup_{t \in [0, \bar{T}]} \left[ \int_0^t |S(t-s)F(s, Y(s))|ds \right]^p \\ &\leq M_{\bar{T}}^p \mathbb{E} \left[ \int_0^{\bar{T}} |F(s, Y(s))|ds \right]^p \\ &\leq (c_1 M_{\bar{T}})^p \mathbb{E} \left[ \int_0^{\bar{T}} [1 + |Y(s)|]ds \right]^p \\ &\leq (c_1 M_{\bar{T}})^p \bar{T}^{p-1} \mathbb{E} \int_0^{\bar{T}} [1 + |Y(s)|]^p ds \\ (3.6) \quad &\leq (c_1 M_{\bar{T}})^p (2\bar{T})^{p-1} \mathbb{E} \int_0^{\bar{T}} [1 + |Y(s)|^p] ds \\ &\leq (c_1 \bar{T} M_{\bar{T}})^p 2^{p-1} [1 + \|Y\|_{p, \bar{T}}^p] < \infty, \quad Y \in \mathcal{E}_p(0, \bar{T}), \end{aligned}$$

here we used the inequality  $(a+b)^p \leq 2^{p-1}(a^p + b^p)$ ,  $a > 0, b > 0$ . Thus,  $\mathcal{Q}_1(\mathcal{E}_p(0, \bar{T})) \subset \mathcal{E}_p(0, \bar{T})$ . Furthermore, due to Proposition 2.4, we have

$$\begin{aligned} \|\mathcal{Q}_2(Y)\|_{p, \bar{T}}^p &= \sup_{t \in [0, \bar{T}]} \mathbb{E} \left| \int_0^t S(t-s)G(s, Y(s))dw_s \right|^p \\ &\leq \left( \frac{p}{p-1} \right)^p c_p(E) \mathbb{E} \left[ \int_0^t |S(t-s)G(s, Y(s))|^2 ds \right]^{\frac{p}{2}} \end{aligned}$$

$$\begin{aligned}
&\leq \left(\frac{pM_{\bar{T}}}{p-1}\right)^p c_p(E) \mathbb{E} \left[ \int_0^{\bar{T}} |G(s, Y(s))|^2 ds \right]^{\frac{p}{2}} \\
&\leq \left(\frac{c_1 p M_{\bar{T}}}{p-1}\right)^p c_p(E) \mathbb{E} \left[ \int_0^{\bar{T}} [1 + |Y(s)|]^2 ds \right]^{\frac{p}{2}} \\
&\leq \left(\frac{c_1 p M_{\bar{T}}}{p-1}\right)^p c_p(E) \bar{T}^{\frac{p-2}{2}} \mathbb{E} \int_0^{\bar{T}} [1 + |Y(s)|]^p ds \\
&\leq \left(\frac{c_1 p M_{\bar{T}}}{p-1}\right)^p c_p(E) \bar{T}^{\frac{p-2}{2}} 2^{p-1} \mathbb{E} \int_0^{\bar{T}} [1 + |Y(s)|^p] ds \\
(3.7) \quad &= \left(\frac{c_1 p M_{\bar{T}}}{p-1}\right)^p c_p(E) \bar{T}^{\frac{p-2}{2}} 2^{p-1} \int_0^{\bar{T}} [1 + \mathbb{E}|Y(s)|^p] ds \\
&\leq \left(\frac{c_1 p M_{\bar{T}}}{p-1}\right)^p c_p(E) \bar{T}^{\frac{p}{2}} 2^{p-1} [1 + \|Y\|_{p, \bar{T}}^p] < \infty, \quad Y \in \mathcal{E}_p(0, \bar{T}).
\end{aligned}$$

Therefore,  $\mathcal{Q}_2(\mathcal{E}_p(0, \bar{T})) \subset \mathcal{E}_p(0, \bar{T})$ . We thus have shown that  $\mathcal{Q}(\mathcal{E}_p(0, \bar{T})) \subset \mathcal{E}_p(0, \bar{T})$ .

Let us next verify that  $\mathcal{Q}$  is a contraction mapping of  $\mathcal{E}_p(0, \bar{T})$ , provided  $\bar{T} > 0$  is sufficiently small. For any  $Y_1, Y_2 \in \mathcal{E}_p(0, \bar{T})$  we have

$$\begin{aligned}
\|\mathcal{Q}_1(Y_1) - \mathcal{Q}_1(Y_2)\|_{p, \bar{T}}^p &= \sup_{t \in [0, \bar{T}]} \mathbb{E} \left| \int_0^t S(t-s)(F(s, Y_1(s)) - F(s, Y_2(s))) ds \right|^p \\
&\leq M_T^p \sup_{t \in [0, \bar{T}]} \mathbb{E} \left[ \int_0^t |F(s, Y_1(s)) - F(s, Y_2(s))| ds \right]^p \\
&\leq (c_2 M_{\bar{T}})^p \mathbb{E} \left[ \int_0^{\bar{T}} |Y_1(s) - Y_2(s)| ds \right]^p \\
&\leq (c_2 M_{\bar{T}})^p \bar{T}^{p-1} \mathbb{E} \int_0^{\bar{T}} |Y_1(s) - Y_2(s)|^p ds \\
&\leq (c_2 \bar{T} M_{\bar{T}})^p \|Y_1 - Y_2\|_{p, \bar{T}}^p
\end{aligned}$$

and

$$\begin{aligned}
&\|\mathcal{Q}_2(Y_1) - \mathcal{Q}_2(Y_2)\|_{p, \bar{T}}^p \\
&= \sup_{t \in [0, \bar{T}]} \mathbb{E} \left| \int_0^t S(t-s)[G(s, Y_1(s)) - G(s, Y_2(s))] dw_s \right|^p \\
&\leq \left(\frac{p}{p-1}\right)^p c_p(E) \sup_{t \in [0, \bar{T}]} \mathbb{E} \left[ \int_0^t |S(t-s)[G(s, Y_1(s)) - G(s, Y_2(s))]|^2 ds \right]^{\frac{p}{2}} \\
&\leq \left(\frac{p M_{\bar{T}}}{p-1}\right)^p c_p(E) \sup_{t \in [0, \bar{T}]} \mathbb{E} \left[ \int_0^t |G(s, Y_1(s)) - G(s, Y_2(s))|^2 ds \right]^{\frac{p}{2}}
\end{aligned}$$

$$\begin{aligned}
&\leq \left(\frac{c_2 p M_{\bar{T}}}{p-1}\right)^p c_p(E) \mathbb{E} \left[ \int_0^{\bar{T}} |Y_1(s) - Y_2(s)|^2 ds \right]^{\frac{p}{2}} \\
&\leq \left(\frac{c_2 p M_{\bar{T}}}{p-1}\right)^p c_p(E) \bar{T}^{\frac{p-2}{2}} \mathbb{E} \int_0^{\bar{T}} |Y_1(s) - Y_2(s)|^p ds \\
&\leq \left(\frac{c_2 p M_{\bar{T}}}{p-1}\right)^p c_p(E) \bar{T}^{\frac{p}{2}} \|Y_1 - Y_2\|_{p, \bar{T}}^p.
\end{aligned}$$

Hence,

$$\begin{aligned}
\|\mathcal{Q}(Y_1) - \mathcal{Q}(Y_2)\|_{p, \bar{T}} &\leq \|\mathcal{Q}_1(Y_1) - \mathcal{Q}_1(Y_2)\|_{p, \bar{T}} + \|\mathcal{Q}_2(Y_1) - \mathcal{Q}_2(Y_2)\|_{p, \bar{T}} \\
&\leq c_2 M_{\bar{T}} \sqrt{\bar{T}} \left[ \sqrt{\bar{T}} + \frac{p C_p^{\frac{1}{p}}(E)}{p-1} \right] \|Y_1 - Y_2\|_{p, \bar{T}}.
\end{aligned}$$

Therefore, if

$$(3.8) \quad c_2 M_{\bar{T}} \sqrt{\bar{T}} \left[ \sqrt{\bar{T}} + \frac{p C_p^{\frac{1}{p}}(E)}{p-1} \right] < 1,$$

then  $\mathcal{Q}$  is contraction in  $\mathcal{E}_p(0, \bar{T})$ .

Since  $\mathcal{Q}$  maps  $\mathcal{E}_p(0, \bar{T})$  into itself and is contraction with respect to the norm of  $\mathcal{E}_p(0, \bar{T})$ ,  $\mathcal{Q}$  has a unique fixed point  $X \in \mathcal{E}_p(0, \bar{T})$ . This shows that  $X(t)$  is a mild solution to (1.1) on  $[0, \bar{T}]$ . In view of (3.8), this solution can be extended on  $[0, T]$  by considering the equation on intervals  $[0, \bar{T}]$ ,  $[\bar{T}, 2\bar{T}]$ ,  $\dots$ . Furthermore, the continuity of  $X$  on  $[0, T]$  follows from the continuity of the stochastic integral (see Proposition 2.4).

**Step 2.** Let us verify uniqueness of the mild solution. Let  $X_1$  and  $X_2$  be mild solutions of (1.1). Then for every  $t \in [0, T]$  we have

$$\begin{aligned}
&\mathbb{E}|X_1(t) - X_2(t)|^2 \\
&= \mathbb{E} \left| \int_0^t S(t-s)[F(s, X_1(s)) - F(s, X_2(s))] ds \right. \\
&\quad \left. + \int_0^t S(t-s)[G(s, X_1(s)) - G(s, X_2(s))] dw_s \right|^2 \\
&\leq 2\mathbb{E} \left| \int_0^t S(t-s)[F(s, X_1(s)) - F(s, X_2(s))] ds \right|^2 \\
&\quad + 2\mathbb{E} \left| \int_0^t S(t-s)[G(s, X_1(s)) - G(s, X_2(s))] dw_s \right|^2 \\
&\leq 2M_{\bar{T}}^2 \mathbb{E} \left[ \int_0^t |F(s, X_1(s)) - F(s, X_2(s))| ds \right]^2
\end{aligned}$$

$$\begin{aligned}
& + 2c(E)\mathbb{E} \int_0^t |S(t-s)[G(s, X_1(s)) - G(s, X_2(s))]|^2 ds \\
& \leq 2c_2^2 M_T^2 \mathbb{E} \left[ \int_0^t |X_1(s) - X_2(s)| ds \right]^2 \\
& \quad + 2c(E) M_T^2 \mathbb{E} \int_0^t |G(s, X_1(s)) - G(s, X_2(s))|^2 ds \\
& \leq 2c_2^2 [t + c(E)] M_T^2 \mathbb{E} \int_0^t |X_1(s) - X_2(s)|^2 ds \\
& \leq 2c_2^2 [T + c(E)] M_T^2 \int_0^t \mathbb{E} |X_1(s) - X_2(s)|^2 ds.
\end{aligned}$$

The Gronwall lemma then derives that  $\mathbb{E}|X_1(t) - X_2(t)|^2 = 0$  for every  $t \in [0, T]$ . Since  $X_1$  and  $X_2$  are continuous, they are indistinguishable.

**Step 3.** Let us finally verify the estimate (3.3). In view of (3.6) and (3.7), we have

$$\begin{aligned}
& \sup_{t \in [0, t]} \mathbb{E}|X(s)|^p \\
& = \|X\|_{p,t}^p = \|\mathcal{Q}(X)\|_{p,t}^p \\
& \leq [\|S(\cdot)\xi\|_{p,t} + \|\mathcal{Q}_1(X)\|_{p,t} + \|\mathcal{Q}_2(X)\|_{p,t}]^p \\
& \leq 3^p [\|S(\cdot)\xi\|_{p,t}^p + \|\mathcal{Q}_1(X)\|_{p,t}^p + \|\mathcal{Q}_2(X)\|_{p,t}^p] \\
& \leq 3^p \left[ M_T^p \mathbb{E}|\xi|^p + (c_1 M_t)^p (2t)^{p-1} \mathbb{E} \int_0^t [1 + |Y(s)|^p] ds \right. \\
& \quad \left. + \left( \frac{c_1 p M_t}{p-1} \right)^p c_p(E) t^{\frac{p-2}{2}} 2^{p-1} \int_0^t [1 + \mathbb{E}|Y(s)|^p] ds \right] \\
& \leq 3^p M_T^p \mathbb{E}|\xi|^p + \left[ (3c_1 M_t)^p (2t)^{p-1} + \left( \frac{3c_1 p M_t}{p-1} \right)^p c_p(E) t^{\frac{p-2}{2}} 2^{p-1} \right] \\
& \quad \times \int_0^t [1 + \sup_{[0,s]} \mathbb{E}|Y(r)|^p] ds \\
& \leq 3^p M_T^p \mathbb{E}|\xi|^p + \left[ (3c_1 M_T)^p (2T)^{p-1} + \left( \frac{3c_1 p M_T}{p-1} \right)^p c_p(E) T^{\frac{p-2}{2}} 2^{p-1} \right] \\
& \quad \times \left[ T + \int_0^t \sup_{[0,s]} \mathbb{E}|Y(r)|^p ds \right], \quad t \in [0, T].
\end{aligned}$$

Then the Gronwall lemma again provides (3.3).  $\square$

**3.2. Existence and regular dependence on initial data of local mild solutions under local linear growth and local Lipschitz conditions.** Let us first

explore existence and uniqueness of local mild solutions of (1.1) under local linear growth and local Lipschitz conditions on  $F$  and  $G$ . Following ideas in [15], we shall construct two lemmas.

LEMMA 3.2. *Let  $(\alpha_1, \alpha_2) \subset \mathbb{R}$ ,  $\Omega_0 \subset \Omega$  and  $\Phi_i \in \mathcal{N}^2(\alpha_1, \alpha_2)$ . If*

$$\mathbf{1}_{\Omega_0} \Phi_1(t) = \mathbf{1}_{\Omega_0} \Phi_2(t) \quad \text{for all } t \in (\alpha_1, \alpha_2),$$

then

$$\mathbf{1}_{\Omega_0} \int_{\alpha_1}^{\alpha_2} \Phi_1(t) dw_t = \mathbf{1}_{\Omega_0} \int_{\alpha_1}^{\alpha_2} \Phi_2(t) dw_t \quad a.s.$$

The proof of Lemma 3.2 for the case  $E = \mathbb{R}$  can be found in [15, Lemma 2.11]. In fact, the arguments are available for any  $M$ -type 2 separable Banach space. So we omit it.

LEMMA 3.3. *Consider two equations of the form (1.1):*

$$(3.9) \quad \begin{cases} dX_i + AX_i dt = F_i(t, X_i) dt + G_i(t, X_i) dw_t, \\ X_i(0) = \xi_i, \end{cases} \quad i = 1, 2.$$

Assume that there exists a constant  $c > 0$  such that

$$|F_i(t, x) - F_i(t, y)| + |G_i(t, x) - G_i(t, y)| \leq c|x - y|$$

and

$$|F_i(t, x)|^2 + |G_i(t, x)|^2 \leq c^2(1 + |x|^2)$$

for  $i = 1, 2, x, y \in E$  and  $t \in [0, T]$ . Suppose further that  $F_1(t, x) = F_2(t, x)$ ,  $G_1(t, x) = G_2(t, x)$  for  $|x| \leq n, 0 \leq t \leq T$  with some  $n > 0$ , and that  $\xi_1 = \xi_2$  for a.s.  $\omega$  for which either  $\xi_1(\omega) < n$  or  $\xi_2(\omega) < n$ . Denote  $\tau_i = \inf\{t : |X_i(t)| > n\}$  with the convention  $\inf \emptyset = T$ . Then

$$\mathbb{P}(\tau_1 = \tau_2) = 1,$$

$$\mathbb{P}\left\{\sup_{0 < t \leq \tau_1} |X_1(t) - X_2(t)| = 0\right\} = 1.$$

PROOF. On the account of Theorem 3.1, there exists a mild solution  $X_i(t)$  of (3.9) on  $[0, T]$ . Consider a function  $\phi : [0, T] \rightarrow \mathbb{R}$  defined by

$$\phi(t) = \begin{cases} 1 & \text{if } |X_1(s)| \leq n \text{ for all } 0 \leq s \leq t, \\ 0 & \text{in all other cases.} \end{cases}$$

Then for  $t \in [0, T]$  we have

$$\begin{cases} \phi(t)(\xi_1 - \xi_2) = 0 & \text{a.s.,} \\ \phi(t)S(t)(\xi_1 - \xi_2) = 0 & \text{a.s.,} \\ \phi(t) = \mathbf{1}_{\{\phi(t)=1\}}\phi(t), \\ \phi(t) = \phi(t)^2. \end{cases}$$

Therefore,

$$\begin{aligned} \phi(t)[X_1(t) - X_2(t)] &= \phi(t) \int_0^t S(t-s)[F_1(s, X_1(s)) - F_2(s, X_1(s))]ds \\ &\quad + \phi(t) \int_0^t S(t-s)[F_2(s, X_1(s)) - F_2(s, X_2(s))]ds \\ &\quad + \phi(t) \int_0^t S(t-s)[G_1(s, X_1(s)) - G_2(s, X_1(s))]dw_s \\ &\quad + \phi(t) \int_0^t S(t-s)[G_2(s, X_1(s)) - G_2(s, X_2(s))]dw_s \\ &= J_1 + J_2 + J_3 + J_4. \end{aligned}$$

When  $\phi(t) = 1$ ,  $F_1(s, X_1(s)) = F_2(s, X_1(s))$  for every  $s \in [0, t]$ . Hence,  $J_1 = 0$ . In addition, we have  $G_1(s, X_1(s)) = G_2(s, X_1(s))$  on  $\{\phi(t) = 1\}$  for all  $s \in [0, t]$ . Lemma 3.2 then provides that

$$\begin{aligned} J_3 &= \mathbf{1}_{\{\phi(t)=1\}}\phi(t) \int_0^t S(t-s)[G_1(s, X_1(s)) - G_2(s, X_1(s))]dw_s \\ &= \mathbf{1}_{\{\phi(t)=1\}} \int_0^t S(t-s)[G_1(s, X_1(s)) - G_2(s, X_1(s))]dw_s \\ &= \mathbf{1}_{\{\phi(t)=1\}} \int_0^t 0dw_s = 0. \end{aligned}$$

Hence,

$$\begin{aligned} \mathbb{E}\phi(t)|X_1(t) - X_2(t)|^2 &= \mathbb{E}|\phi(t)[X_1(t) - X_2(t)]|^2 = \mathbb{E}|J_2 + J_4|^2 \\ (3.10) \quad &\leq 2\mathbb{E}|J_2|^2 + 2\mathbb{E}|J_4|^2. \end{aligned}$$



Let us estimate  $\mathbb{E}|J_2|^2$  and  $\mathbb{E}|J_4|^2$ . Since  $\phi(t)$  decreases in  $t$ , we have

$$\begin{aligned}
2|J_2|^2 &\leq 2 \left| \int_0^t \phi(s) S(t-s) [F_2(s, X_1(s)) - F_2(s, X_2(s))] ds \right|^2 \\
&\leq 2t \int_0^t \phi(s) |S(t-s) [F_2(s, X_1(s)) - F_2(s, X_2(s))]|^2 ds \\
&\leq 2tM_t^2 \int_0^t \phi(s) |F_2(s, X_1(s)) - F_2(s, X_2(s))|^2 ds \\
&\leq 2tc^2M_t^2 \int_0^t \phi(s) |X_1(s) - X_2(s)|^2 ds, \quad t \in [0, T].
\end{aligned}$$

Thus,

$$(3.11) \quad \mathbb{E}|J_2|^2 \leq 2Tc^2M_T^2 \int_0^t \mathbb{E}\phi(s) |X_1(s) - X_2(s)|^2 ds, \quad t \in [0, T].$$

On the other hand, we have  $\phi(s) = 1$  on  $\{\phi(t) = 1\}$  for every  $s \in [0, t]$ . Lemma 3.2 again provides that

$$\begin{aligned}
2|J_4|^2 &= 2 \left| \phi(t) \mathbf{1}_{\{\phi(t)=1\}} \int_0^t S(t-s) [G_2(s, X_1(s)) - G_2(s, X_2(s))] dw_s \right|^2 \\
&= 2 \left| \mathbf{1}_{\{\phi(t)=1\}} \int_0^t S(t-s) [G_2(s, X_1(s)) - G_2(s, X_2(s))] dw_s \right|^2 \\
&= 2 \left| \mathbf{1}_{\{\phi(t)=1\}} \int_0^t \phi(s) S(t-s) [G_2(s, X_1(s)) - G_2(s, X_2(s))] dw_s \right|^2 \\
&\leq 2 \left| \int_0^t \phi(s) S(t-s) [G_2(s, X_1(s)) - G_2(s, X_2(s))] dw_s \right|^2.
\end{aligned}$$

Using Proposition 2.4, we then obtain that

$$\begin{aligned}
2\mathbb{E}|J_4|^2 &\leq 2c(E) \mathbb{E} \int_0^t |\phi(s) S(t-s) [G_2(s, X_1(s)) - G_2(s, X_2(s))]|^2 ds \\
&\leq 2c(E)M_t^2 \mathbb{E} \int_0^t \phi(s) |G_2(s, X_1(s)) - G_2(s, X_2(s))|^2 ds \\
(3.12) \quad &\leq 2c^2c(E)M_T^2 \int_0^t \mathbb{E}\phi(s) |X_1(s) - X_2(s)|^2 ds, \quad t \in [0, T].
\end{aligned}$$

Substituting (3.11) and (3.12) into (3.10), we observe that

$$\begin{aligned}
&\mathbb{E}\phi(t) |X_1(t) - X_2(t)|^2 \\
&\leq 2c^2[T + c(E)]M_T^2 \int_0^t \mathbb{E}\phi(s) |X_1(s) - X_2(s)|^2 ds, \quad t \in [0, T].
\end{aligned}$$

Thanks to the Gronwall lemma, we verify that  $\mathbb{E}\phi(t)|X_1(t) - X_2(t)|^2 = 0$  for every  $t \in [0, T]$ . As a consequence,

$$\phi(t)|X_1(t) - X_2(t)|^2 = 0 \quad \text{a.s.}$$

From the definition of  $\phi$ , it is then clear that  $X_1(t) = X_2(t)$  a.s.  $t \in (0, \tau_1]$  and  $\mathbb{P}(\tau_2 \geq \tau_1) = 1$ . Similarly, we have  $X_1(t) = X_2(t)$  a.s.  $t \in (0, \tau_2]$  and  $\mathbb{P}(\tau_1 \geq \tau_2) = 1$ . We thus have shown that

$$\mathbb{P}\{X_1(t) = X_2(t)\} = \mathbb{P}(\tau_1 = \tau_2) = 1, \quad t \in (0, \tau_1].$$

In addition, by the continuity of  $X_1(t)$  and  $X_2(t)$ , we conclude that

$$\sup_{0 < t \leq \tau_1} |X_1(t) - X_2(t)|^2 = 0 \quad \text{a.s.}$$

It completes the proof.  $\square$

**THEOREM 3.4** (local existence). *Suppose that for any  $n > 0$  there exist  $c_n > 0$  and  $\bar{c}_n > 0$  such that whenever  $|x| \leq n$ ,  $|y| \leq n$  and  $t \in [0, T]$ , the following two conditions hold:*

(i) *The local growth condition*

$$|F(t, x)| + |G(t, x)| \leq \bar{c}_n(1 + |x|).$$

(ii) *The local Lipschitz condition*

$$(3.13) \quad |F(t, x) - F(t, y)| + |G(t, x) - G(t, y)| \leq c_n|x - y|.$$

*Then there exists a unique maximal local mild solution  $\{X(t), t \in [0, \tau)\}$  to (1.1). Furthermore, there exists a constant  $\alpha = \alpha(\bar{c}_n, M_T, T) > 0$  depending only on  $\bar{c}_n, M_T$  and  $T$  such that*

$$(3.14) \quad \mathbb{E}|X(t \wedge \tau_n)|^2 \leq \alpha(1 + \mathbb{E}|\xi|^2), \quad t \geq 0, n = 0, 1, \dots,$$

*where  $\{\tau_n\}_{n=0}^\infty$  is a sequence of stopping times defined by*

$$\tau_n = \inf\{t \in [0, T] : |X(t)| > n\}$$

*with the convention  $\inf \emptyset = T$  and  $\tau_0 = 0$  a.s.*

PROOF. Let us first show existence of a maximal local mild solution by using the truncation method. For  $n = 1, 2, \dots$ , we denote

$$F_n(t, x) = \begin{cases} F(t, x) & \text{if } |x| \leq n, \\ F(t, x)(2 - \frac{|x|}{n}) & \text{if } n < |x| \leq 2n, \\ 0 & \text{if } |x| > 2n, \end{cases}$$

$$G_n(t, x) = \begin{cases} G(t, x) & \text{if } |x| \leq n, \\ G(t, x)(2 - \frac{|x|}{n}) & \text{if } n < |x| \leq 2n, \\ 0 & \text{if } |x| > 2n, \end{cases}$$

and

$$\xi_{n_0} = \begin{cases} \xi & \text{if } |\xi| \leq n, \\ 0 & \text{if } |\xi| > n. \end{cases}$$

It is easy to see that the measurable functions  $F_n$  and  $G_n$  satisfy the linear growth and global Lipschitz conditions. On the account of Theorem 3.1, there exists a unique mild solution  $X_n(t)$  on  $[0, T]$  of the system

$$(3.15) \quad \begin{cases} dX_n + AX_n dt = F_n(t, X_n)dt + G_n(t, X_n)dw_t, \\ X_n(0) = \xi_{n_0}. \end{cases}$$

Define the stopping times

$$(3.16) \quad \tau_n = \inf\{t \in [0, T] : |X_n(t)| > n\}, \quad n = 1, 2, \dots$$

By Lemma 3.3, we observe that

$$X_n(t) = X_m(t) \quad \text{a.s. if } 0 \leq t \leq \tau_n \text{ and } m > n.$$

Hence, the sequence  $\{\tau_n\}_n$  increases and has a limit  $\tau = \lim_{n \rightarrow \infty} \tau_n \leq T$  a.s. We then define  $\{X(t), 0 \leq t < \tau\}$  by

$$(3.17) \quad X(t) = X_n(t), \quad t \in [\tau_{n-1}, \tau_n], n \geq 1.$$

It is clear that  $X(t)$  is continuous and  $X \in \mathcal{N}^2(0, t \wedge \tau_n)$  for every  $t \geq 0, n \geq 1$ . In addition, on  $\{\tau < T\}$  we have

$$\liminf_{t \rightarrow \tau} |X(t)| \geq \liminf_{n \rightarrow \infty} |X(\tau_n)| = \liminf_{n \rightarrow \infty} |X_n(\tau_n)| = \infty.$$

On the other hand, using Lemma 3.2, we obtain that

$$X(t \wedge \tau_n) = X_n(t \wedge \tau_n)$$

$$\begin{aligned}
&= S(t \wedge \tau_n) \xi_{n_0} + \int_0^{t \wedge \tau_n} S(t \wedge \tau_n - s) F_n(s, X_n(s)) ds \\
&\quad + \int_0^{t \wedge \tau_n} S(t \wedge \tau_n - s) G_n(s, X_n(s)) dw_s \\
(3.18) \quad &= S(t \wedge \tau_n) \xi_{n_0} + \int_0^{t \wedge \tau_n} S(t \wedge \tau_n - s) F(s, X(s)) ds \\
&\quad + \int_0^{t \wedge \tau_n} S(t \wedge \tau_n - s) G(s, X(s)) dw_s.
\end{aligned}$$

Therefore,  $\{X(t), t \in [0, \tau)\}$  is a maximal local mild solution of (1.1).

Let us now verify the estimate (3.14). Using the same argument as in the proof of the estimate (3.3) in Theorem 3.1 to the stochastic integral equation (3.18), we conclude that

$$\mathbb{E}|X(t \wedge \tau_n)|^2 \leq \alpha(1 + \mathbb{E}|\xi_{n_0}|^2) \leq \alpha(1 + \mathbb{E}|\xi|^2), \quad t \geq 0, n \geq 1,$$

where  $\alpha = \alpha(\bar{c}_n, M_T, T) > 0$  is some constant depending only on  $\bar{c}_n, M_T$  and  $T$ . Clearly, by (3.16) and (3.17),

$$\tau_n = \inf\{t \in [0, T] : |X(t)| > n\}.$$

The estimate (3.14) thus has been verified.

Let us finally verify uniqueness of the solution. Let  $\{\bar{X}(t), t \in [0, \bar{\tau})\}$  be another maximal local mild solution of (1.1). Denote

$$\bar{\tau}_n = \inf\{t \in [0, \bar{\tau}) : |\bar{X}(t)| > n\} \quad \text{and} \quad \tau_n^* = \tau_n \wedge \bar{\tau}_n.$$

Then the sequence  $\{\tau_n^*\}_n$  increases and converges to  $\tau \wedge \bar{\tau}$  a.s. as  $n \rightarrow \infty$ . For  $t \geq 0$  and  $n = 1, 2, \dots$ , we have

$$\begin{aligned}
&\mathbb{E}|X(t \wedge \tau_n^*) - \bar{X}(t \wedge \tau_n^*)|^2 \\
&\leq 2\mathbb{E}\left|\int_0^{t \wedge \tau_n^*} S(t \wedge \tau_n^* - s) \{F(s, X(s)) - F(s, \bar{X}(s))\} ds\right|^2 \\
&\quad + 2\mathbb{E}\left|\int_0^{t \wedge \tau_n^*} S(t \wedge \tau_n^* - s) \{G(s, X(s)) - G(s, \bar{X}(s))\} dw_s\right|^2 \\
&\leq 2TM_T^2 \mathbb{E} \int_0^{t \wedge \tau_n^*} |F(s, X(s)) - F(s, \bar{X}(s))|^2 ds \\
&\quad + 2c(E) \mathbb{E} \int_0^{t \wedge \tau_n^*} |S(t \wedge \tau_n^* - s) \{G(s, X(s)) - G(s, \bar{X}(s))\}|^2 ds \\
&\leq 2Tc_n^2 M_T^2 \mathbb{E} \int_0^{t \wedge \tau_n^*} |X(s) - \bar{X}(s)|^2 ds
\end{aligned}$$

$$\begin{aligned}
& + 2c(E)M_T^2\mathbb{E}\int_0^{t\wedge\tau_n^*}|G(s, X(s)) - G(s, \bar{X}(s))|^2 ds \\
& \leq 2(T + c(E))c_n^2 M_T^2\mathbb{E}\int_0^{t\wedge\tau_n^*}|X(s) - \bar{X}(s)|^2 ds \\
& = 2(T + c(E))c_n^2 M_T^2\mathbb{E}\int_0^t \mathbf{1}_{[0, \tau_n^*]}(s)|X(s \wedge \tau_n^*) - \bar{X}(s \wedge \tau_n^*)|^2 ds \\
(3.19) \quad & \leq 2(T + c(E))c_n^2 M_T^2\int_0^t \mathbb{E}|X(s \wedge \tau_n^*) - \bar{X}(s \wedge \tau_n^*)|^2 ds.
\end{aligned}$$

The Gronwall lemma then gives that  $\mathbb{E}|X(t \wedge \tau_n^*) - \bar{X}(t \wedge \tau_n^*)|^2 = 0$ . Hence,  $X(t \wedge \tau_n^*) = \bar{X}(t \wedge \tau_n^*)$  for every  $n \geq 1$  and  $t \geq 0$ . Letting  $n \rightarrow \infty$ , we conclude that  $X(t) = \bar{X}(t)$  for every  $t \in [0, \tau \wedge \bar{\tau}]$ .

On the other hand, if  $\mathbb{P}\{\tau < \bar{\tau}\} < 1$  then for almost sure  $\omega \in \{\tau < \bar{\tau}\}$ ,  $\bar{X}(t, \omega)$  is continuous at  $\tau(\omega)$ . We then arrive at a contradiction

$$\infty > |\bar{X}(\tau(\omega), \omega)| = \lim_{n \rightarrow \infty} |\bar{X}(\tau_n(\omega), \omega)| = \lim_{n \rightarrow \infty} |X(\tau_n(\omega), \omega)| = \infty.$$

Therefore,  $\mathbb{P}\{\tau < \bar{\tau}\} = 1$ . Similarly, we have  $\mathbb{P}\{\tau > \bar{\tau}\} = 1$ . We thus have shown that  $\mathbb{P}\{\tau = \bar{\tau}\} = 1$ . The theorem is proved.  $\square$

Let us now study dependence of the maximal local mild solution on the initial data. It turns out that the maximal local mild solution  $\{X(t), t \in [0, \tau)\}$  of (1.1), which is shown in Theorem 3.4, depends continuously on the initial data in the sense specified in the following theorem.

**THEOREM 3.5.** *Let the assumption on the functions  $F$  and  $G$  of Theorem 3.4 be satisfied. Let  $\{X(t), t \in [0, \tau)\}$  and  $\{\bar{X}(t), t \in [0, \bar{\tau})\}$  be the maximal local solutions of (1.1) with the initial values  $\xi$  and  $\bar{\xi}$ , respectively. Then there exist two positive constants  $C_1 = C_1(c_n, M_T, T)$  and  $C_2 = C_2(c_n, \bar{c}_n, M_T, T)$  satisfying the estimates*

$$(3.20) \quad \mathbb{E}|X(t \wedge \tau_n) - \bar{X}(t \wedge \tau_n)|^2 \leq C_1 \mathbb{E}|\xi - \bar{\xi}|^2$$

and

$$\begin{aligned}
(3.21) \quad & \mathbb{E}|X(t \wedge \tau_n) - X(s \wedge \tau_n)|^2 \\
& \leq C_2[\mathbb{E}|S(t \wedge \tau_n - s \wedge \tau_n) - I|X(s \wedge \tau_n)|^2 + (1 + \mathbb{E}|\xi|^2)(t - s)]
\end{aligned}$$

for every  $0 \leq s \leq t \leq T$ , where  $\{\tau_n\}_{n=0}^\infty$  is a sequence of stopping times defined by

$$\tau_n = \inf\{t \in [0, T] : |X(t)| > n \text{ or } |\bar{X}(t)| > n\}.$$

PROOF. Let us first prove (3.20). The case  $\mathbb{E}|\xi - \bar{\xi}|^2 = \infty$  is obvious. Hence, we may assume that  $\mathbb{E}|\xi - \bar{\xi}|^2 < \infty$ . In view of the proof of Theorem 3.4 (see (3.18)), we have

$$(3.22) \quad \begin{aligned} X(t \wedge \tau_n) = & S(t \wedge \tau_n)\xi + \int_0^{t \wedge \tau_n} S(t \wedge \tau_n - r)F(r, X(r))dr \\ & + \int_0^{t \wedge \tau_n} S(t \wedge \tau_n - r)G(r, X(r))dw_r, \end{aligned}$$

and

$$\begin{aligned} \bar{X}(t \wedge \tau_n) = & S(t \wedge \tau_n)\bar{\xi} + \int_0^{t \wedge \tau_n} S(t \wedge \tau_n - r)F(r, \bar{X}(r))dr \\ & + \int_0^{t \wedge \tau_n} S(t \wedge \tau_n - r)G(r, \bar{X}(r))dw_r. \end{aligned}$$

Similarly to (3.19), we obtain that

$$\begin{aligned} & \mathbb{E}|X(t \wedge \tau_n) - \bar{X}(t \wedge \tau_n)|^2 \\ & \leq 3\mathbb{E}|S(t \wedge \tau_n)(\xi - \bar{\xi})|^2 \\ & \quad + 3(T + c(E))c_n^2 M_T^2 \int_0^t \mathbb{E}|X(r \wedge \tau_n) - \bar{X}(r \wedge \tau_n)|^2 dr \\ & \leq 3M_T^2 \mathbb{E}|\xi - \bar{\xi}|^2 \\ & \quad + 3(T + c(E))c_n^2 M_T^2 \int_0^t \mathbb{E}|X(r \wedge \tau_n) - \bar{X}(r \wedge \tau_n)|^2 dr, \quad t \in [0, T]. \end{aligned}$$

The Gronwall lemma then provides (3.20).

Let us now verify (3.21). By the semigroup property, from (3.22) we observe that

$$\begin{aligned} & X(t \wedge \tau_n) - X(s \wedge \tau_n) \\ & = S(t \wedge \tau_n - s \wedge \tau_n) \left[ S(s \wedge \tau_n)\xi + \int_0^{s \wedge \tau_n} S(s \wedge \tau_n - r)F(r, X(r))dr \right. \\ & \quad \left. + \int_0^{s \wedge \tau_n} S(s \wedge \tau_n - r)G(r, X(r))dw_r \right] + \int_{s \wedge \tau_n}^{t \wedge \tau_n} S(t \wedge \tau_n - r)F(r, X(r))dr \\ & \quad + \int_{s \wedge \tau_n}^{t \wedge \tau_n} S(t \wedge \tau_n - r)G(r, X(r))dw_r - X(s \wedge \tau_n) \\ & = [S(t \wedge \tau_n - s \wedge \tau_n) - I]X(s \wedge \tau_n) + \int_{s \wedge \tau_n}^{t \wedge \tau_n} S(t \wedge \tau_n - r)F(r, X(r))dr \end{aligned}$$

$$+ \int_{s \wedge \tau_n}^{t \wedge \tau_n} S(t \wedge \tau_n - r) G(r, X(r)) dw_r.$$

Using the local growth condition on  $F$  and  $G$ , we then observe that

$$\begin{aligned} & \mathbb{E}|X(t \wedge \tau_n) - X(s \wedge \tau_n)|^2 \\ & \leq 3\mathbb{E}|[S(t \wedge \tau_n - s \wedge \tau_n) - I]X(s \wedge \tau_n)|^2 \\ & \quad + 3\mathbb{E}\left|\int_{s \wedge \tau_n}^{t \wedge \tau_n} S(t \wedge \tau_n - r) F(r, X(r)) dr\right|^2 \\ & \quad + 3\mathbb{E}\left|\int_{s \wedge \tau_n}^{t \wedge \tau_n} S(t \wedge \tau_n - r) G(r, X(r)) dw_r\right|^2 \\ & \leq 3\mathbb{E}|[S(t \wedge \tau_n - s \wedge \tau_n) - I]X(s \wedge \tau_n)|^2 \\ & \quad + 3\mathbb{E}(t \wedge \tau_n - s \wedge \tau_n) \int_{s \wedge \tau_n}^{t \wedge \tau_n} |S(t \wedge \tau_n - r)|^2 |F(r, X(r))|^2 dr \\ & \quad + 3c(E)\mathbb{E} \int_{s \wedge \tau_n}^{t \wedge \tau_n} |S(t \wedge \tau_n - r)|^2 |G(r, X(r))|^2 dr \\ & \leq 3\mathbb{E}|[S(t \wedge \tau_n - s \wedge \tau_n) - I]X(s \wedge \tau_n)|^2 \\ & \quad + 6c_n^2 M_T^2 T \mathbb{E} \int_{s \wedge \tau_n}^{t \wedge \tau_n} [1 + |X(r)|^2] dr \\ & \quad + 6c_n^2 M_T^2 c(E) \mathbb{E} \int_{s \wedge \tau_n}^{t \wedge \tau_n} [1 + |X(r)|^2] dr \\ & = 3\mathbb{E}|[S(t \wedge \tau_n - s \wedge \tau_n) - I]X(s \wedge \tau_n)|^2 \\ & \quad + 6c_n^2 M_T^2 [T + c(E)] \int_s^t [1 + \mathbb{E}|X(r \wedge \tau_n)|^2] dr. \end{aligned}$$

By virtue of (3.14), we obtain that

$$\begin{aligned} & \mathbb{E}|X(t \wedge \tau_n) - X(s \wedge \tau_n)|^2 \\ & \leq 3\mathbb{E}|[S(t \wedge \tau_n - s \wedge \tau_n) - I]X(s \wedge \tau_n)|^2 \\ & \quad + 6c_n^2 M_T^2 [T + c(E)](1 + \alpha + \alpha\mathbb{E}|\xi|^2)(t - s), \end{aligned}$$

where  $\alpha = \alpha(\bar{c}_n, M_T, T)$  is some positive constant. Thus, (3.21) has been verified.  $\square$

**3.3. Existence and regular dependence on initial data of global mild solutions under linear growth and local Lipschitz conditions.** Let us first show existence and uniqueness of mild solutions of (1.1) under linear growth and local Lipschitz conditions on  $F$  and  $G$ .

**THEOREM 3.6** (global existence). *Suppose that  $F$  and  $G$  satisfies the linear growth condition (3.1) and the local Lipschitz condition (3.13) and that  $\mathbb{E}|\xi|^2 < \infty$ . Then*

(i) *there exists a unique mild solution  $X(t)$  of (1.1) on  $[0, T]$  such that*

$$(3.23) \quad \mathbb{E}|X(t)|^2 \leq \alpha_1(1 + \mathbb{E}|\xi|^2), \quad t \in [0, T],$$

*where  $\alpha_1$  is some constant depending only on  $c_1, M_T$  and  $T$ .*

(ii) *For any  $p > 2$  there exists a constant  $\alpha_2 > 0$  depending only on  $c_1, p, M_T$  and  $T$  such that*

$$(3.24) \quad \mathbb{E} \sup_{0 \leq t \leq T} |X(t)|^p \leq \alpha_2(1 + \mathbb{E}|\xi|^p).$$

**PROOF.**

• Proof of (i).

It is already known by Theorem 3.4 that there exists a unique maximal local mild solution  $\{X(t), t \in [0, \tau)\}$  to (1.1) such that

$$\begin{cases} \mathbb{E}|X(t \wedge \tau_k)|^2 \leq \alpha_1(1 + \mathbb{E}|\xi|^2), & t \geq 0, k \geq 1, \\ X(\tau_k) = k, & k \geq 1, \end{cases}$$

where  $\alpha_1 > 0$  is some constant depending only on  $c_1, M_T$  and  $T$ .

Let us first verify that  $\tau = T$  a.s. Indeed, if this statement is false, then there would exist  $t_0 \in (0, T)$  and  $\epsilon \in (0, 1)$  such that  $\mathbb{P}\{\tau < t_0\} > \epsilon$ . Hence, by denoting  $\Omega_k = \{\tau_k \leq t_0\}$ , there exists  $k_0 \in \mathbb{N} \setminus \{0\}$  such that  $\mathbb{P}(\Omega_k) \geq \epsilon$  for all  $k \geq k_0$ . Since the sequence  $\{\Omega_k\}_k$  is decreasing, we observe that  $\mathbb{P}(\cap_{k=k_0} \Omega_k) \geq \epsilon$ . From this, for every  $k \geq k_0$  we have

$$\begin{aligned} \alpha_1(1 + \mathbb{E}|\xi|^2) &\geq \mathbb{E}|X(t_0 \wedge \tau_k)|^2 \\ &\geq \mathbb{E}|\mathbf{1}_{\cap_{k=k_0} \Omega_k} X(t_0 \wedge \tau_k)|^2 \\ &\geq \mathbb{E}|\mathbf{1}_{\cap_{k=k_0} \Omega_k} X(\tau_k)|^2 \\ &= k^2 \mathbb{P}\{\cap_{k=k_0} \Omega_k\} = \epsilon k^2. \end{aligned}$$

Letting  $k \rightarrow \infty$ , we arrive at a contradiction:  $\infty > \alpha_1(1 + \mathbb{E}|\xi|^2) = \infty$ . Thus,  $\tau = T$  a.s.

Let us now show that  $X(t)$  is defined and is continuous at  $t = T$  and that  $X$  satisfies the estimate (3.23). Since  $\tau = T$  a.s., we have

$$(3.25) \quad X(t) = S(t)\xi + \mathcal{Q}_1(X)(t) + \mathcal{Q}_2(X)(t), \quad t \in [0, T],$$



where  $\mathcal{Q}_1$  and  $\mathcal{Q}_2$  are defined by (3.4) and (3.5), respectively. Using the same argument as in the proof of the estimate (3.3) of Theorem 3.1 to the stochastic integral equation (3.25), we observe that

$$(3.26) \quad \mathbb{E}|X(t)|^2 \leq \alpha_1(1 + \mathbb{E}|\xi|^2), \quad t \in [0, T],$$

where  $\alpha_1$  is some constant depending only on  $c_1, M_T$  and  $T$ . Clearly, the estimate (3.26) derives that  $\mathcal{Q}_1(X)(t)$  and  $\mathcal{Q}_2(X)(t)$  are defined at  $t = T$ . Furthermore, Proposition 2.4 provides the continuity of  $\mathcal{Q}_2(X)$  at  $t = T$ . The continuity of  $\mathcal{Q}_1(X)$  at  $t = T$  can be seen easily from (3.26) and the equality

$$\begin{aligned} & \mathcal{Q}_1(X)(T) - \mathcal{Q}_1(X)(t) \\ &= \int_0^T S(T-t)S(t-s)F(s, X(s))ds - \int_0^t S(t-s)F(s, X(s))ds \\ &= [S(T-t) - I] \int_0^t S(t-s)F(s, X(s))ds + \int_t^T S(T-s)F(s, X(s))ds \end{aligned}$$

with any  $t \in [0, T]$ . Therefore, by setting

$$X(T) = S(T)\xi + \mathcal{Q}_1(X)(T) + \mathcal{Q}_2(X)(T),$$

we conclude that the obtained process  $\{X(t), t \in [0, T]\}$  is a unique mild solution of (1.1) on  $[0, T]$ . Furthermore, the estimate (3.23) follows from (3.26) and the continuity of  $X(t)$  on  $[0, T]$ .

• Proof of (ii).

The case  $\mathbb{E}|\xi|^p = \infty$  is obvious. Therefore, we can assume that  $\mathbb{E}|\xi|^p < \infty$ . To simply the proof, we shall use a notation  $C$  to denote positive constants which are determined by the constants  $c_1, p, M_T$  and  $T$ . So, it may change from occurrence to occurrence. Since

$$X(t) = S(t)\xi + \mathcal{Q}_1(X)(t) + \mathcal{Q}_2(X)(t),$$

for every  $s \in [0, T]$  we have

$$\begin{aligned} |X(s)|^p &\leq C \left[ |\xi|^p + \left\{ \int_0^s |S(s-u)F(u, X(u))| du \right\}^p + |\mathcal{Q}_2(X)(s)|^p \right] \\ &\leq C \left[ |\xi|^p + \left\{ \int_0^s |F(u, X(u))| du \right\}^p + |\mathcal{Q}_2(X)(s)|^p \right] \\ &\leq C \left[ |\xi|^p + \left\{ \int_0^s (1 + |X(u)|) du \right\}^p + |\mathcal{Q}_2(X)(s)|^p \right] \\ &\leq C \left[ |\xi|^p + \int_0^s (1 + |X(u)|^p) du + |\mathcal{Q}_2(X)(s)|^p \right]. \end{aligned}$$

Hence,

$$\begin{aligned} & \mathbb{E} \sup_{0 \leq s \leq t} |X(s)|^p \\ & \leq C \left[ \mathbb{E} |\xi|^p + \int_0^t (1 + \mathbb{E} \sup_{0 \leq u \leq s} |X(u)|^p) ds + \mathbb{E} \sup_{0 \leq s \leq t} |\mathcal{Q}_2(X)(s)|^p \right]. \end{aligned}$$

On the other hand, applying the Burkholder-Davis-Gundy inequality [18, Theorem 3.28] to the continuous martingale  $\{\mathcal{Q}_2(X)(s), s \in [0, T]\}$ , we see that

$$\begin{aligned} \mathbb{E} \sup_{0 \leq s \leq t} |\mathcal{Q}_2(X)(s)|^p & \leq C \mathbb{E} \left[ \int_0^t |S(s-u)G(u, X(u))|^2 du \right]^{\frac{p}{2}} \\ & \leq C \mathbb{E} \left[ \int_0^t |G(u, X(u))|^2 du \right]^{\frac{p}{2}} \\ & \leq C \mathbb{E} \left[ \int_0^t (1 + |X(u)|^2) du \right]^{\frac{p}{2}} \\ & \leq C \int_0^t [1 + \mathbb{E} \sup_{0 \leq u \leq s} |X(u)|^p] ds. \end{aligned}$$

Therefore, we have shown that

$$\mathbb{E} \sup_{0 \leq s \leq t} |X(s)|^p \leq C \left[ 1 + \mathbb{E} |\xi|^p + \int_0^t \mathbb{E} \sup_{0 \leq u \leq s} |X(u)|^p ds \right].$$

By the Gronwall lemma, we conclude that

$$\mathbb{E} \sup_{0 \leq s \leq T} |X(s)|^p \leq C(1 + \mathbb{E} |\xi|^p).$$

The proof is complete.  $\square$

In the remain of the subsection, we will give the continuous dependence of the global mild solution on the initial data.

**THEOREM 3.7.** *Let (3.1) and (3.13) be satisfied. Let  $X$  and  $\bar{X}$  be the global solutions of (1.1) with the initial values  $\xi$  and  $\bar{\xi}$  satisfying the condition  $\mathbb{E} |\xi|^2 + \mathbb{E} |\bar{\xi}|^2 < \infty$ , respectively. Then there exist two positive constants  $C_1 = C_1(c_1, M_T, T)$  and  $C_2 = C_2(c_1, \bar{c}_n, M_T, T)$  such that*

$$\mathbb{E} |X(t) - \bar{X}(t)|^2 \leq C_1 \mathbb{E} |\xi - \bar{\xi}|^2$$

and

$$\mathbb{E} |X(t) - X(s)|^2 \leq C_2 [\mathbb{E} |S(t-s) - I|X(s)|^2 + (1 + \mathbb{E} |\xi|^2)(t-s)]$$

for every  $0 \leq s \leq t \leq T$ .

As the proof of this theorem is similar to that of Theorem 3.5, we may omit it.

**4. Linear evolution equations with additive noise.** This section handles linear evolution equations with additive noise: the functions  $F$  and  $G$  are considered to be dependent only on  $t$ , i.e.  $F(t, X) = F(t)$  and  $G(t, X) = G(t)$ . Let us rewrite the equation (1.1) in the form

$$(4.1) \quad \begin{cases} dX + AXdt = F(t)dt + G(t)dw_t, \\ X(0) = \xi. \end{cases}$$

We shall explore existence of strict and mild solutions and regularity of solutions of (4.1).

Throughout this section, we suppose that

- the spectrum  $\sigma(A)$  of  $A$  is contained in an open sectorial domain  $\Sigma_\varpi$ :

$$(4.2) \quad \sigma(A) \subset \Sigma_\varpi = \{\lambda \in \mathbb{C} : |\arg \lambda| < \varpi\}, \quad 0 < \varpi < \frac{\pi}{2},$$

- there exists a constant  $M_\varpi > 0$  such that

$$(4.3) \quad \|(\lambda - A)^{-1}\| \leq \frac{M_\varpi}{|\lambda|}, \quad \lambda \notin \Sigma_\varpi.$$

- the non-random functions  $F: [0, T] \rightarrow E$  and  $G: [0, T] \rightarrow E$  satisfy one of the following two conditions:

$$(4.4) \quad F \in \mathcal{F}^{\beta, \sigma}((0, T]; E), G \in \mathcal{F}^{\beta + \frac{1}{2}, \sigma}((0, T]; E)$$

$$\text{with } 0 < \sigma < \beta \leq \frac{1}{2},$$

$$(4.5) \quad F \in \mathcal{F}^{\beta, \sigma} \text{ with } 0 < \sigma < \beta \leq 1 \text{ and } G \in \mathcal{B}([0, T]; E).$$

The following lemma presents some useful results. The proof can be found in the monograph [30].

LEMMA 4.1. *The linear operator  $A$  satisfies the following properties.*

- (i)  $(-A)$  generates an analytic semigroup  $S(t) = e^{-tA}$ .
- (ii)

$$(4.6) \quad |A^\nu S(t)| \leq \iota_\nu t^{-\nu}, \quad t \in (0, T], \nu \in [0, \infty),$$

where  $\iota_\nu = \sup_{0 < t \leq T} t^\nu |A^\nu S(t)| < \infty$ . In particular,

$$(4.7) \quad |S(t)| \leq \iota_0, \quad t \in [0, T].$$

(iii) *There exists  $\nu > 0$  such that*

$$(4.8) \quad |S(t)| \leq \iota_0 e^{-\nu t}, \quad t \in [0, T].$$

(iv) *For every  $\theta \in (0, 1]$*

$$(4.9) \quad |[S(t) - I]A^{-\theta}| \leq \frac{\iota_0^{1-\theta}}{\theta} t^\theta, \quad t \in [0, T].$$

(v)

$$(4.10) \quad AS(\cdot)U \in \mathcal{F}^{\beta, \sigma}((0, T]; E) \quad \text{for every } U \in \mathcal{D}(A^\beta).$$

4.1. *Existence of strict solutions.* This subsection presents existence of strict solutions of the autonomous linear evolution equation (4.1).

THEOREM 4.2. *Assume that*

$$(4.11) \quad \begin{aligned} &\text{there exist } \delta \in (0, \frac{1}{2}) \text{ and } c_\delta > 0 \text{ such that } |AS(t)| < c_\delta t^{-\delta} \\ &\text{for every } t \in (0, T]. \end{aligned}$$

*Then there exists a unique strict solution of (4.1) in the space  $\mathcal{C}((0, T]; \mathcal{D}(A))$ .*

PROOF. Let us observe that the integral  $\int_0^t S(t-s)G(s)dw_s$  is well-defined and is continuous. Indeed, it suffices to show that  $\int_0^t |S(t-s)G(s)|^2 ds$  is finite for  $t \in (0, T]$ . If (4.4) takes place then from (2.5) and (4.7), we have

$$\begin{aligned} \int_0^t |S(t-s)G(s)|^2 ds &\leq \int_0^t \iota_0^2 |G|_{\mathcal{F}^{\beta+\frac{1}{2}, \sigma}}^2 s^{2\beta-1} ds \\ &= \frac{\iota_0^2 |G|_{\mathcal{F}^{\beta+\frac{1}{2}, \sigma}}^2 t^{2\beta}}{2\beta} < \infty, \quad t \in [0, T]. \end{aligned}$$

Otherwise, if (4.5) takes place then from (4.7), we have

$$\int_0^t |S(t-s)G(s)|^2 ds \leq \iota_0^2 t |G|_{\mathcal{B}((0, T]; E)}^2 < \infty, \quad t \in [0, T].$$

On the other hand, by (2.5) and (4.8), we have

$$(4.12) \quad \int_0^t |S(t-s)F(s)| ds \leq \iota_0 |F|_{\mathcal{F}^{\beta, \sigma}} \int_0^t e^{-\nu(t-s)} s^{\beta-1} ds \leq \frac{\iota_0 |F|_{\mathcal{F}^{\beta, \sigma}} t^\beta}{\beta}.$$

Hence,  $\int_0^t S(t-s)F(s)ds$  is continuous on  $[0, T]$ . We thus have shown that (4.1) has a unique mild solution  $X(t) = I_1(t) + I_2(t)$ , where

$$(4.13) \quad \begin{aligned} I_1(t) &= S(t)\xi + \int_0^t S(t-s)F(s)ds, \\ I_2(t) &= \int_0^t S(t-s)G(s)dw_s. \end{aligned}$$

We shall show that this mild solution is a strict solution in the space  $\mathcal{C}((0, T]; \mathcal{D}(A))$ . For this purpose, we divide the proof into two steps.

**Step 1.** Let us verify that  $I_1 \in \mathcal{C}((0, T]; \mathcal{D}(A))$  and satisfies the integral equation

$$(4.14) \quad I_1(t) + \int_0^t AI_1(s)ds = \xi + \int_0^t F(s)ds, \quad t \in (0, T].$$

Let  $A_n = A(1 + \frac{A}{n})^{-1}$ ,  $n \in \mathbb{N} \setminus \{0\}$  be the Yosida approximation of  $A$ . Then  $A_n$  satisfies (4.2) and (4.3) uniformly and generates an analytic semigroup  $S_n(t)$  (see e.g. [30]). Furthermore, for every  $\nu \in [0, \infty)$  we have

$$(4.15) \quad \lim_{n \rightarrow \infty} A_n^\nu S_n(t) = A^\nu S(t) \quad (\text{limit in } \mathcal{L}(E))$$

and there exists  $\varsigma_\nu > 0$  independent of  $n$  such that for every  $t \in (0, T]$

$$(4.16) \quad \begin{cases} |A_n^\nu S_n(t)| \leq \varsigma_\nu t^{-\nu} & \text{if } \nu > 0, \\ |A_n^\nu S_n(t)| \leq \varsigma_\nu e^{-\varsigma_\nu t} & \text{if } \nu = 0. \end{cases}$$

Consider a function

$$I_{1n}(t) = S_n(t)\xi + \int_0^t S_n(t-s)F(s)ds, \quad t \in [0, T].$$

Due to (4.15) and (4.16),  $\lim_{n \rightarrow \infty} I_{1n}(t) = I_1(t)$  a.s. Since  $A_n$  is bounded, we verify that

$$dI_{1n} = [-A_n I_{1n} + F(t)]dt, \quad t \in (0, T].$$

From this equation, for any  $\epsilon \in (0, T)$  we obtain that

$$(4.17) \quad I_{1n}(t) = I_{1n}(\epsilon) + \int_\epsilon^t [F(s) - A_n I_{1n}(s)]ds, \quad t \in [\epsilon, T].$$

We shall observe convergence of  $A_n I_{1n}$ . We have

$$A_n I_{1n}(t)$$

$$\begin{aligned}
&= A_n S_n(t) \xi + \int_0^t A_n S_n(t-s) [F(s) - F(t)] ds + \int_0^t A_n S_n(t-s) ds F(t) \\
(4.18) \quad &= A_n S_n(t) \xi + \int_0^t A_n S_n(t-s) [F(s) - F(t)] ds + [I - S_n(t)] F(t).
\end{aligned}$$

Using (2.5) and (4.16), we observe that

$$\begin{aligned}
&|A_n I_{1n}(t)| \\
&\leq \varsigma_1 t^{-1} |\xi| + \int_0^t \varsigma_1 |F|_{\mathcal{F}^{\beta, \sigma}}(t-s)^{\sigma-1} s^{\beta-\sigma-1} ds + (1 + \varsigma_0 e^{-\varsigma_0 t}) |F|_{\mathcal{F}^{\beta, \sigma}} t^{\beta-1} \\
(4.19) \quad &= \varsigma_1 |\xi| t^{-1} + [1 + \varsigma_1 \mathbf{B}(\beta - \sigma, \sigma) + \varsigma_0 e^{-\varsigma_0 t}] |F|_{\mathcal{F}^{\beta, \sigma}} t^{\beta-1}, \quad t \in (0, T],
\end{aligned}$$

where  $\mathbf{B}(\cdot, \cdot)$  is the beta function. Furthermore, due to (4.15) and (4.18),

$$\lim_{n \rightarrow \infty} A_n I_{1n}(t) = W(t),$$

where

$$W(t) = AS(t) \xi + \int_0^t AS(t-s) [F(s) - F(t)] ds + [I - S(t)] F(t).$$

Let us verify that  $W(t)$  is continuous on  $(0, T]$ . Indeed, let  $t_0 \in (0, T]$ . Using (2.5) and (4.6), for every  $t \geq t_0$  we have

$$\begin{aligned}
&|W(t) - W(t_0)| \\
&\leq |AS(t_0)[S(t-t_0) - I] \xi| + |[I - S(t)]F(t) - [I - S(t_0)]F(t_0)| \\
&\quad + \left| \int_{t_0}^t AS(t-s) [F(s) - F(t)] ds + \int_0^{t_0} AS(t-s) ds [F(t_0) - F(t)] \right. \\
&\quad + \int_0^{t_0} S(t-t_0) AS(t_0-s) [F(s) - F(t_0)] ds \\
&\quad \left. - \int_0^{t_0} AS(t_0-s) [F(s) - F(t_0)] ds \right| \\
&\leq \iota_1 t_0^{-1} |S(t-t_0) \xi - \xi| + |[I - S(t)]F(t) - [I - S(t_0)]F(t_0)| \\
&\quad + \int_{t_0}^t |AS(t-s)| |F(s) - F(t)| ds + |[S(t-t_0) - S(t)] [F(t_0) - F(t)]| \\
&\quad + \int_0^{t_0} |[S(t-t_0) - I] AS(t_0-s) [F(s) - F(t_0)]| ds \\
&\leq \iota_1 t_0^{-1} |S(t-t_0) \xi - \xi| + |[I - S(t)]F(t) - [I - S(t_0)]F(t_0)|
\end{aligned}$$

$$\begin{aligned}
& + \int_{t_0}^t \iota_1 |F|_{\mathcal{F}^{\beta, \sigma}}(t-s)^{\sigma-1} s^{\beta-\sigma-1} ds + |S(t-t_0) - S(t)| |F(t_0) - F(t)| \\
& + |S(t-t_0) - I| \int_0^{t_0} \iota_1 |F|_{\mathcal{F}^{\beta, \sigma}}(t_0-s)^{\sigma-1} s^{\beta-\sigma-1} ds \\
& \leq \iota_1 t_0^{-1} |S(t-t_0)\xi - \xi| + |[I - S(t)]F(t) - [I - S(t_0)]F(t_0)| \\
& + \varsigma_1 |F|_{\mathcal{F}^{\beta, \sigma}} t_0^{\beta-\sigma-1} \int_{t_0}^t (t-s)^{\sigma-1} ds + |S(t-t_0) - S(t)| |F(t_0) - F(t)| \\
& + \iota_1 |F|_{\mathcal{F}^{\beta, \sigma}} \mathbf{B}(\beta - \sigma, \sigma) t_0^{\beta-1} |S(t-t_0) - I| \\
(4.20) \quad & \leq \iota_1 t_0^{-1} |S(t-t_0)\xi - \xi| + |[I - S(t)]F(t) - [I - S(t_0)]F(t_0)| \\
& + \frac{\iota_1 |F|_{\mathcal{F}^{\beta, \sigma}}}{\sigma} t_0^{\beta-\sigma-1} (t-t_0)^{\sigma} + |S(t-t_0) - S(t)| |F(t_0) - F(t)| \\
& + \iota_1 |F|_{\mathcal{F}^{\beta, \sigma}} \mathbf{B}(\beta - \sigma, \sigma) t_0^{\beta-1} |S(t-t_0) - I|.
\end{aligned}$$

Thus,  $\lim_{t \searrow t_0} W(t) = W(t_0)$ . Similarly, we obtain that  $\lim_{t \nearrow t_0} W(t) = W(t_0)$ . Hence,  $W(t)$  is continuous at  $t = t_0$  and then at every point in  $(0, T]$ .

By the above arguments, we have

$$I_1(t) = \lim_{n \rightarrow \infty} I_{1n}(t) = \lim_{n \rightarrow \infty} A_n^{-1} A_n I_{1n}(t) = A^{-1} W(t).$$

This shows that  $I_1(t) \in \mathcal{D}(A)$  and  $AI_1(t) = W(t) \in \mathcal{C}((0, T]; E)$ .

Let us prove that  $\int_0^t W(s) ds$  exists. Indeed, by virtue of (2.5), (4.6) and (4.7), we have

$$\begin{aligned}
\int_0^t |[I - S(r)]F(r)| dr & \leq (1 + \iota_0) |F|_{\mathcal{F}^{\beta, \sigma}} \int_0^t r^{\beta-1} dr \\
& = \frac{(1 + \iota_0) |F|_{\mathcal{F}^{\beta, \sigma}} t^{\beta}}{\beta}, \quad t \in [0, T],
\end{aligned}$$

and

$$\begin{aligned}
& \int_0^t \left| \int_0^s AS(s-r)[F(r) - F(s)] dr \right| ds \\
& \leq \int_0^t \int_0^s |AS(s-r)| |F(r) - F(s)| dr ds \\
& \leq \iota_1 |F|_{\mathcal{F}^{\beta, \sigma}} \int_0^t \int_0^s (s-r)^{\sigma-1} r^{\beta-\sigma-1} dr ds \\
& = \frac{\iota_1 |F|_{\mathcal{F}^{\beta, \sigma}} \mathbf{B}(\beta - \sigma, \sigma) t^{\beta}}{\beta}, \quad t \in [0, T].
\end{aligned}$$

These estimates show that  $\int_0^t [I - S(r)]F(r)dr$  and  $\int_0^t \int_0^s AS(s-r)[F(r) - F(s)]drds$  exist for  $t \in [0, T]$ . Therefore, the existence of  $\int_0^t W(s)ds$  for  $t \in [0, T]$  follows from the equality

$$\begin{aligned} \int_0^t W(s)ds &= \int_0^t AS(r)\xi dr + \int_0^t \int_0^s AS(s-r)[F(r) - F(s)]drds \\ &\quad + \int_0^t [I - S(r)]F(r)dr \\ &= [I - S(t)]\xi + \int_0^t \int_0^s AS(s-r)[F(r) - F(s)]drds \\ &\quad + \int_0^t [I - S(r)]F(r)dr. \end{aligned}$$

In view of (4.19), by applying the Lebesgue dominate convergence theorem to (4.17), for any  $\epsilon \in (0, T)$  we obtain that

$$(4.21) \quad I_1(t) = I_1(\epsilon) + \int_\epsilon^t [F(s) - AI_1(s)]ds, \quad t \in [\epsilon, T].$$

From (4.12) and (4.13),  $\lim_{\epsilon \rightarrow 0} I_1(\epsilon) = \xi$ . Letting  $\epsilon \rightarrow 0$  in (4.21), we then observe the equation (4.14).

**Step 2.** Let us observe that  $I_2 \in \mathcal{C}([0, T]; \mathcal{D}(A))$  and satisfies the equation

$$(4.22) \quad I_2(t) + \int_0^t AI_2(s)ds = \int_0^t G(u)dw_u, \quad t \in (0, T].$$

If (4.4) takes place, then by using (2.5) and (4.11), we have

$$\begin{aligned} \int_0^t |AS(t-s)G(s)|^2 ds &\leq \int_0^t c_\delta^2 (t-s)^{-2\delta} |G|_{\mathcal{F}^{\beta+\frac{1}{2}, \sigma}}^2 s^{2\beta-1} ds \\ &= c_\delta^2 |G|_{\mathcal{F}^{\beta+\frac{1}{2}, \sigma}}^2 t^{2(\beta-\delta)} \mathbf{B}(2\beta, 1-2\delta) < \infty, \quad t \in (0, T]. \end{aligned}$$

Otherwise, if (4.5) takes place then

$$\begin{aligned} \int_0^t |AS(t-s)G(s)|^2 ds &\leq c_\delta^2 \int_0^t (t-s)^{-2\delta} |G|_{\mathcal{B}((0, T]; E)}^2 ds \\ &= \frac{c_\delta^2 t^{1-2\delta} |G|_{\mathcal{B}((0, T]; E)}^2}{1-2\delta} < \infty, \quad t \in (0, T]. \end{aligned}$$

Hence, the integral  $\int_0^t AS(t-s)G(s)dw_s, t \in (0, T]$  is well-defined and then is continuous. Since  $A$  is closed, we obtain that

$$AI_2(t) = \int_0^t AS(t-s)G(s)dw_s, \quad t \in (0, T].$$



Using the Fubini formula, we have

$$\begin{aligned}
A \int_0^t I_2(s) ds &= \int_0^t \int_0^s AS(s-u)G(u)dw_u ds \\
&= \int_0^t \int_u^t AS(s-u)G(u)ds dw_u \\
&= \int_0^t [G(u) - S(t-u)G(u)]dw_u \\
&= \int_0^t G(u)dw_u - \int_0^t S(t-u)G(u)dw_u \\
&= \int_0^t G(u)dw_u - I_2(t), \quad t \in (0, T].
\end{aligned}$$

This means that  $I_2$  satisfies (4.22).

From these two steps, we conclude that  $X(t)$  is a strict solution in the space  $\mathcal{C}((0, T]; \mathcal{D}(A))$ .  $\square$

REMARK 4.3. In **Step 1** of the proof of Theorem 4.2, the assumption (4.11) is not used. Using this assumption, we can reduce the proof of the statement of this step. However, we do not present such a proof, because it is not useful for the study of regularity of mild solutions in the next theorems. In fact, this assumption is only to guarantee the existence of the integral  $\int_0^t AS(t-s)G(s)dw_s, t \in (0, T]$ .

4.2. *Regularity of mild solutions.* In this subsection, we will study regularity of mild solutions of (4.1) without the condition (4.11) in Theorem 4.2. The initial value is considered in the domain  $\mathcal{D}(A^\beta)$ .

THEOREM 4.4. *Suppose that  $\xi \in \mathcal{D}(A^\beta)$  a.s. Then there exists a mild solution  $X$  of (4.1) in the space*

$$X \in \mathcal{C}([0, T]; \mathcal{D}(A^\beta)) \cap \mathcal{C}^\beta([0, T]; E).$$

Furthermore,  $X$  satisfies the estimates

$$(4.23) \quad \mathbb{E}|X(t)|^2 \leq \rho_1[\mathbb{E}|\xi|^2 + |F|_{\mathcal{F}^{\beta, \sigma}}^2 + |G|_{\mathcal{F}^{\beta+\frac{1}{2}, \sigma}}^2],$$

when (4.4) takes place, and

$$(4.24) \quad \mathbb{E}|X(t)|^2 \leq \rho_2[\mathbb{E}|\xi|^2 + |F|_{\mathcal{F}^{\beta, \sigma}}^2 + (1 - e^{-\rho_2 t})|G|_{\mathcal{B}([0, t]; E)}^2],$$

when (4.5) takes place. Here,  $\rho_1$  and  $\rho_2$  are two positive constants depending only on  $A, \beta$  and  $\sigma$ .

PROOF. The existence of the mild solution  $X(t) = I_1(t) + I_2(t)$  of (4.1) is already shown in Theorem 4.2, where  $I_1(t)$  and  $I_2(t)$  are defined by (4.13). First, let us assume that the condition (4.4) takes place. The proof for this case is divided into several steps.

**Step 1.** Let us show that  $I_1(t) \in \mathcal{D}(A^\beta)$  for  $t \in [0, T]$  and that  $A^\beta I_1 \in \mathcal{C}([0, T]; E)$ . Since  $S(t)$  is strongly continuous, we have

$$\lim_{t \rightarrow s} |A^\beta S(t)\xi - A^\beta S(s)\xi| = \lim_{t \rightarrow s} |[S(t) - S(s)]A^\beta \xi| = 0, \quad s \in [0, T].$$

Therefore,  $A^\beta S(t)\xi$  is continuous on  $[0, T]$ . Because of

$$A^\beta I_1(t) = A^\beta S(t)\xi + A^\beta \int_0^t S(t-s)F(s)ds,$$

it suffices to show that  $A^\beta \int_0^t S(t-s)F(s)ds$  is well-defined and is continuous on  $[0, T]$ .

Let us verify the first assertion. Using the inequalities (2.5) and (4.6), we have

$$\begin{aligned} \int_0^t |A^\beta S(t-s)F(s)|ds &\leq \int_0^t |A^\beta S(t-s)||F(s)|ds \\ &\leq |F|_{\mathcal{F}^{\beta, \sigma} \iota_\beta} \int_0^t (t-s)^{-\beta} s^{\beta-1} ds \\ &= |F|_{\mathcal{F}^{\beta, \sigma} \iota_\beta} \int_0^1 u^{\beta-1} (1-u)^{-\beta} du \\ (4.25) \quad &= \iota_\beta |F|_{\mathcal{F}^{\beta, \sigma}} \mathbf{B}(\beta, 1-\beta), \quad t \in [0, T]. \end{aligned}$$

Hence,  $\int_0^t A^\beta S(t-s)F(s)ds$  is well-defined. Since  $A^\beta$  is closed, we obtain that

$$A^\beta \int_0^t S(t-s)F(s)ds = \int_0^t A^\beta S(t-s)F(s)ds.$$

Let us now verify the second assertion, i.e. to verify the continuity of  $A^\beta \int_0^t S(t-s)F(s)ds$  on  $[0, T]$ . Fix  $t_0 \in [0, T]$ . We consider two cases.

Case 1:  $t_0 > 0$ . For every  $t \geq t_0$  we have

$$\begin{aligned} &\left| \int_0^t A^\beta S(t-s)F(s)ds - \int_0^{t_0} A^\beta S(t_0-s)F(s)ds \right| \\ &\leq \left| \int_0^{t_0} A^\beta [S(t-s) - S(t_0-s)]F(s)ds \right| + \left| \int_{t_0}^t A^\beta S(t-s)F(s)ds \right| \\ &\leq \int_0^{t_0} |A^\beta S(t_0-s)||F(s)|ds |S(t-t_0) - I| + \int_{t_0}^t |A^\beta S(t-s)||F(s)|ds. \end{aligned}$$

Using (2.5) and (4.6), we observe that

$$\begin{aligned}
& \left| \int_0^t A^\beta S(t-s)F(s)ds - \int_0^{t_0} A^\beta S(t_0-s)F(s)ds \right| \\
& \leq \int_0^{t_0} \iota_\beta(t_0-s)^{-\beta} |F|_{\mathcal{F}^{\beta,\sigma}} s^{\beta-1} ds |S(t-t_0) - I| + \int_{t_0}^t \iota_\beta(t-s)^{-\beta} \sup_{s \in [t_0, t]} |F(s)| ds \\
& = |F|_{\mathcal{F}^{\beta,\sigma}} \iota_\beta \mathbf{B}(\beta, 1-\beta) |S(t-t_0) - I| + \frac{\iota_\beta \sup_{s \in [t_0, t]} |F(s)| (t-t_0)^{1-\beta}}{1-\beta} \\
& \rightarrow 0 \text{ as } t \searrow t_0.
\end{aligned}$$

This means that  $\int_0^t A^\beta S(t-s)F(s)ds$  is right-continuous at  $t = t_0$ . Similarly, we can show that it is left-continuous at  $t = t_0$ . Therefore,  $A^\beta \int_0^t S(t-s)F(s)ds$  is continuous at  $t = t_0$ .

Case 2:  $t_0 = 0$ . By the property of the space  $\mathcal{F}^{\beta,\sigma}((0, T]; E)$ , we may put  $z = \lim_{t \searrow 0} t^{1-\beta} F(t)$ . We have

$$\begin{aligned}
& \left| A^\beta \int_0^t S(t-s)F(s)ds \right| \\
& = \left| \int_0^t A^\beta S(t-s)[F(s) - F(t)]ds \right| + \left| \int_0^t A^\beta S(t-s)F(t)ds \right| \\
& = \left| \int_0^t A^\beta S(t-s)[F(s) - F(t)]ds \right| + \left| [I - S(t)]A^{\beta-1}F(t) \right| \\
& \leq \int_0^t |A^\beta S(t-s)| |F(t) - F(s)| ds + |t^{\beta-1}[I - S(t)]A^{\beta-1}[t^{1-\beta}F(t) - z]| \\
& \quad + |t^{\beta-1}[I - S(t)]A^{\beta-1}z|.
\end{aligned}$$

Using (2.4), (4.6) and (4.9), we obtain that

$$\begin{aligned}
& \limsup_{t \searrow 0} \left| A^\beta \int_0^t S(t-s)F(s)ds \right| \\
& \leq \iota_\beta \limsup_{t \searrow 0} \int_0^t \iota_\beta(t-s)^{-\beta} |F(t) - F(s)| ds \\
& \quad + \frac{\iota_\beta}{1-\beta} \limsup_{t \searrow 0} |t^{1-\beta}F(t) - z| \\
& \quad + \limsup_{t \searrow 0} |t^{\beta-1}[I - S(t)]A^{\beta-1}z| \\
& = \iota_\beta \limsup_{t \searrow 0} \int_0^t (t-s)^{\sigma-\beta} s^{-1+\beta-\sigma} \frac{s^{1-\beta+\sigma} |F(t) - F(s)|}{(t-s)^\sigma} ds
\end{aligned}$$

$$\begin{aligned}
& + \frac{\iota_\beta}{1-\beta} \limsup_{t \searrow 0} |t^{1-\beta} F(t) - z| \\
& + \limsup_{t \searrow 0} |t^{\beta-1} [I - S(t)] A^{\beta-1} z| \\
& \leq \iota_\beta \mathbf{B}(\beta - \sigma, 1 - \beta + \sigma) \limsup_{t \searrow 0} \sup_{s \in [0, t]} \frac{s^{1-\beta+\sigma} |F(t) - F(s)|}{(t-s)^\sigma} \\
& + \limsup_{t \searrow 0} |t^{\beta-1} [I - S(t)] A^{\beta-1} z| \\
& = \limsup_{t \searrow 0} |t^{\beta-1} [I - S(t)] A^{\beta-1} z|.
\end{aligned}$$

Since  $\mathcal{D}(A^\beta)$  is dense in  $E$ , there exists a sequence  $\{z_n\}_n$  in  $\mathcal{D}(A^\beta)$  that converges to  $z$  as  $n \rightarrow \infty$ . Using (4.9), we observe that

$$\begin{aligned}
& \limsup_{t \searrow 0} \left| A^\beta \int_0^t S(t-s) F(s) ds \right| \\
& \leq \limsup_{t \searrow 0} |t^{\beta-1} [I - S(t)] A^{\beta-1} (z - z_n)| + \limsup_{t \searrow 0} |t^{\beta-1} [I - S(t)] A^{-1} A^\beta z_n| \\
& \leq \frac{\iota_\beta}{1-\beta} |z - z_n| + \iota_0 \limsup_{t \searrow 0} t^\beta |A^\beta z_n| \\
& = \frac{\iota_\beta}{1-\beta} |z - z_n|, \quad n = 1, 2, \dots
\end{aligned}$$

Letting  $n \rightarrow \infty$ , we obtain that  $\lim_{t \searrow 0} A^\beta \int_0^t S(t-s) F(s) ds = 0$ . This means that  $A^\beta \int_0^t S(t-s) F(s) ds$  is right-continuous at  $t = t_0$ .

From these two cases, we conclude that  $A^\beta \int_0^t S(t-s) F(s) ds$  is continuous at  $t = t_0$  and then at every point in  $[0, T]$ . The second assertion has been shown.

**Step 2.** Let us verify that  $A^\beta X \in \mathcal{C}([0, T]; E)$ . In view of (2.5) and (4.6), we have

$$\begin{aligned}
(4.26) \quad \int_0^t |A^\beta S(t-s) G(s)|^2 ds & \leq \iota_\beta^2 |G|_{\mathcal{F}^{\beta+\frac{1}{2}, \sigma}}^2 \int_0^t (t-s)^{-2\beta} s^{2\beta-1} ds \\
& = \iota_\beta^2 |G|_{\mathcal{F}^{\beta+\frac{1}{2}, \sigma}}^2 \mathbf{B}(2\beta, 1-2\beta) < \infty.
\end{aligned}$$

Thus,  $\int_0^t A^\beta S(t-s) G(s) dw_s$  is well-defined. Since  $A^\beta$  is closed, we obtain that

$$A^\beta I_2(t) = \int_0^t A^\beta S(t-s) G(s) dw_s.$$

Thanks to Proposition 2.4,  $A^\beta I_2$  is a continuous square integrable martingale on  $[0, T]$ . On the account of **Step 1**, we conclude that

$$A^\beta X = A^\beta I_1 + A^\beta I_2 \in \mathcal{C}([0, T]; E).$$

**Step 3.** Let us show that  $I_1$  is  $\beta$ -Hölder continuous on  $[0, T]$ . From (4.6), (4.7), (4.13) and (4.14), for every  $0 \leq s < t \leq T$  we observe that

$$\begin{aligned}
 |I_1(t) - I_1(s)| &= \left| \int_s^t F(u) du - \int_s^t AI_1(u) du \right| \\
 &= \left| \int_s^t F(u) du - \int_s^t AS(u) \xi du - \int_s^t \int_0^u AS(u-r) F(r) dr du \right| \\
 &\leq \left| \int_s^t [F(u) - AS(u) \xi] du \right| + \int_s^t \left| \int_0^u AS(u-r) F(u) dr \right| du \\
 &\quad + \int_s^t \left| \int_0^u AS(u-r) [F(u) - F(r)] dr \right| du \\
 (4.27) \quad &\leq \left| \int_s^t [F(u) - AS(u) \xi] du \right| + \int_s^t |[I - S(u)] F(u)| du \\
 &\quad + \int_s^t \int_0^u |AS(u-r)| |F(u) - F(r)| dr du \\
 &\leq \int_s^t [|F(u) - AS(u) \xi| + (1 + \iota_0) |F(u)|] du \\
 &\quad + \iota_1 \int_s^t \int_0^u (u-r)^{-1} |F(u) - F(r)| dr du \\
 (4.28) \quad &= I_{11}(s, t) + I_{12}(s, t).
 \end{aligned}$$

We shall give estimates for  $I_{11}$  and  $I_{12}$ . Since  $\xi \in \mathcal{D}(A^\beta)$  a.s., we have  $AS(\cdot) \xi \in \mathcal{F}^{\beta, \sigma}((0, T]; E)$  a.s. (see (4.10)). Therefore,

$$F(\cdot) - AS(\cdot) \xi \in \mathcal{F}^{\beta, \sigma}((0, T]; E) \quad \text{a.s.}$$

In view of (2.5), we see that

$$\begin{aligned}
 I_{11}(s, t) &\leq \int_s^t [|F(\cdot) - AS(\cdot) \xi|_{\mathcal{F}^{\beta, \sigma}} u^{\beta-1} + |F|_{\mathcal{F}^{\beta, \sigma}} (1 + \iota_0) u^{\beta-1}] du \\
 &= \frac{|F(\cdot) - AS(\cdot) \xi|_{\mathcal{F}^{\beta, \sigma}} + |F|_{\mathcal{F}^{\beta, \sigma}} (1 + \iota_0)}{\beta} (t^\beta - s^\beta) \\
 &\leq \frac{|F(\cdot) - AS(\cdot) \xi|_{\mathcal{F}^{\beta, \sigma}} + |F|_{\mathcal{F}^{\beta, \sigma}} (1 + \iota_0)}{\beta} (t - s)^\beta.
 \end{aligned}$$

Meanwhile, using (2.5), we have

$$\begin{aligned}
 I_{12}(s, t) &= \iota_1 \int_s^t \int_0^u (u-r)^{\sigma-1} r^{\beta-1-\sigma} \frac{r^{1-\beta+\sigma} |F(u) - F(r)|}{(u-r)^\sigma} dr du \\
 &\leq \iota_1 |F|_{\mathcal{F}^{\beta, \sigma}} \int_s^t \int_0^u (u-r)^{\sigma-1} r^{\beta-\sigma-1} dr du
 \end{aligned}$$

$$\begin{aligned}
&= \iota_1 |F|_{\mathcal{F}^{\beta, \sigma}} \int_s^t u^{\beta-1} \int_0^1 (1-v)^{\sigma-1} v^{\beta-\sigma-1} dv du \\
&= \iota_1 |F|_{\mathcal{F}^{\beta, \sigma}} \mathbf{B}(\beta - \sigma, \sigma) \int_s^t u^{\beta-1} du \\
&= \frac{\iota_1 |F|_{\mathcal{F}^{\beta, \sigma}} \mathbf{B}(\beta - \sigma, \sigma)}{\beta} (t^\beta - s^\beta) \\
(4.29) \quad &\leq \frac{\iota_1 |F|_{\mathcal{F}^{\beta, \sigma}} \mathbf{B}(\beta - \sigma, \sigma)}{\beta} (t - s)^\beta.
\end{aligned}$$

Thus,  $I_1(\cdot)$  is  $\beta$ -Hölder continuous on  $[0, T]$ .

**Step 4.** Let us show that  $X$  is  $\beta$ -Hölder continuous on  $[0, T]$ . On the account of **Step 3**, it suffices to show that  $I_2 \in \mathcal{C}^\beta([0, T]; E)$ . Let  $0 \leq s < t \leq T$ , then

$$I_2(t) - I_2(s) = \int_s^t S(t-r)G(r)dw_r + \int_0^s [S(t-r) - S(s-r)]G(r)dw_r.$$

Since the integrals in the right hand side are independent and have zero expectation, we have

$$\begin{aligned}
&\mathbb{E}|I_2(t) - I_2(s)|^2 \\
&= \mathbb{E} \left| \int_s^t S(t-r)G(r)dw_r \right|^2 + \mathbb{E} \left| \int_0^s [S(t-r) - S(s-r)]G(r)dw_r \right|^2 \\
&\leq c(E) \left[ \int_s^t |S(t-r)|^2 |G(r)|^2 dr + \int_0^s |S(t-r) - S(s-r)|^2 |G(r)|^2 dr \right].
\end{aligned}$$

Using (2.5), (4.6) and (4.7), we then observe that

$$\begin{aligned}
&\mathbb{E}|I_2(t) - I_2(s)|^2 \\
&\leq c(E) |G|_{\mathcal{F}^{\beta+\frac{1}{2}, \sigma}}^2 \left[ \iota_0^2 \int_s^t r^{2\beta-1} dr + \int_0^s \left| \int_{s-r}^{t-r} AS(u)du \right|^2 r^{2\beta-1} dr \right] \\
&\leq c(E) |G|_{\mathcal{F}^{\beta+\frac{1}{2}, \sigma}}^2 \left[ \frac{\iota_0^2 (t^{2\beta} - s^{2\beta})}{2\beta} + \iota_1^2 \int_0^s \left( \int_{s-r}^{t-r} u^{-1} du \right)^2 r^{2\beta-1} dr \right].
\end{aligned}$$

Dividing  $-1$  as  $-1 = -\beta + \beta - 1$ , we have

$$\begin{aligned}
&\left( \int_{s-r}^{t-r} u^{-1} du \right)^2 \leq \left[ \int_{s-r}^{t-r} (s-r)^{-\beta} u^{\beta-1} du \right]^2 \\
&= (s-r)^{-2\beta} \frac{[(t-r)^\beta - (s-r)^\beta]^2}{\beta^2} \\
(4.30) \quad &\leq (s-r)^{-2\beta} \frac{(t-s)^{2\beta}}{\beta^2}.
\end{aligned}$$

Hence,

$$\begin{aligned}
& \mathbb{E}|I_2(t) - I_2(s)|^2 \\
& \leq c(E)|G|_{\mathcal{F}^{\beta+\frac{1}{2},\sigma}}^2 \left[ \frac{\iota_0^2(t^{2\beta} - s^{2\beta})}{2\beta} + \frac{\iota_1^2(t-s)^{2\beta}}{\beta^2} \int_0^s (s-r)^{-2\beta} r^{2\beta-1} dr \right] \\
(4.31) \quad & \leq c(E)|G|_{\mathcal{F}^{\beta+\frac{1}{2},\sigma}}^2 \left[ \frac{\iota_0^2}{2\beta} + \frac{\iota_1^2 \mathbf{B}(2\beta, 1-2\beta)}{\beta^2} \right] (t-s)^{2\beta}.
\end{aligned}$$

In addition, by the definition of the stochastic integral,  $I_2(t)$  is a Gaussian process. Theorem 2.10 then provides that  $I_2 \in \mathcal{C}^\beta([0, T]; E)$ .

**Step 5.** Let us prove (4.23). Taking  $s = 0, t \in (0, T]$  in (4.27), it yields that

$$\begin{aligned}
& |I_1(t) - I_1(0)| \\
& \leq \left| \int_0^t [F(u) - AS(u)\xi] du \right| + \int_0^t |[I - S(u)]F(u)| du \\
& \quad + \int_0^t \int_0^u |AS(u-r)| |F(u) - F(r)| dr du \\
& \leq \int_0^t [1 + |I - S(u)|] |F(u)| du + \left| \int_0^t AS(u)\xi du \right| \\
& \quad + c_1 \int_0^t \int_0^u (u-r)^{-1} |F(u) - F(r)| dr du.
\end{aligned}$$

In view of (2.5), (4.7), (4.28) and (4.29), we have

$$\begin{aligned}
& |I_1(t) - I_1(0)| \\
& \leq \int_0^t (2 + \iota_0) |F(u)| du + |[I - S(t)]\xi| + I_{12}(0, t) \\
& \leq (2 + \iota_0) \int_0^t |F|_{\mathcal{F}^{\beta,\sigma}} u^{\beta-1} du + (1 + \iota_0) |\xi| + \frac{\iota_1 |F|_{\mathcal{F}^{\beta,\sigma}} \mathbf{B}(\beta - \sigma, \sigma)}{\beta} t^\beta \\
& = \frac{[2 + \iota_0 + \iota_1 \mathbf{B}(\beta - \sigma, \sigma)] t^\beta |F|_{\mathcal{F}^{\beta,\sigma}}}{\beta} + (1 + \iota_0) |\xi|.
\end{aligned}$$

Then

$$|I_1(t)| \leq (2 + \iota_0) |\xi| + \frac{[2 + \iota_0 + \iota_1 \mathbf{B}(\beta - \sigma, \sigma)] t^\beta |F|_{\mathcal{F}^{\beta,\sigma}}}{\beta}, \quad t \in (0, T].$$

Obviously, this inequality also holds true for  $t = 0$ . As a consequence, there exists  $c_1 > 0$  depending only on  $A, \beta$  and  $\sigma$  such that

$$(4.32) \quad \mathbb{E}|I_1(t)|^2 \leq c_1 [\mathbb{E}|\xi|^2 + |F|_{\mathcal{F}^{\beta,\sigma}}^2], \quad t \in [0, T].$$

On the other hand, using (2.5) and (4.8), we have

$$\begin{aligned}
 \mathbb{E}|I_2(t)|^2 &\leq c(E) \int_0^t |S(t-s)G(s)|^2 ds \\
 &\leq c(E) \iota_0^2 |G|_{\mathcal{F}^{\beta+\frac{1}{2},\sigma}}^2 \int_0^t e^{-2\nu(t-s)} s^{2\beta-1} ds \\
 (4.33) \quad &\leq c(E) c_{\nu,\beta} \iota_0^2 |G|_{\mathcal{F}^{\beta+\frac{1}{2},\sigma}}^2, \quad t \in [0, T],
 \end{aligned}$$

where  $c_{\nu,\beta} = \sup_{t \in [0,\infty)} \int_0^t e^{-2\nu(t-s)} s^{2\beta-1} ds < \infty$ .

Combining (4.32) and (4.33), we conclude that

$$\begin{aligned}
 \mathbb{E}|X(t)|^2 &\leq 2\mathbb{E}[I_1(t)^2 + I_2(t)^2] \\
 &\leq 2c_1[\mathbb{E}|\xi|^2 + |F|_{\mathcal{F}^{\beta,\sigma}}^2] + 2c(E) c_{\nu,\beta} \iota_0^2 |G|_{\mathcal{F}^{\beta+\frac{1}{2},\sigma}}^2,
 \end{aligned}$$

from which it follows (4.23).

From these steps, the assertion of the theorem under the condition (4.4) is proved.

Let us now assume that the condition (4.5) takes place. Similarly to the above case, we will verify that the assertions in **Steps 1-4** still hold true. Indeed, the assertions in **Step 1** and **Step 3** are obviously true, since they are independent of the conditions (4.4) and (4.5).

To show the assertion in **Step 2**, it suffices to verify that  $\int_0^t A^\beta S(t-s)G(s)ds$  is well-defined. This follows from a fact that

$$\begin{aligned}
 \int_0^t |A^\beta S(t-s)G(s)|^2 ds &\leq \iota_\beta^2 |G|_{\mathcal{B}([0,t];E)}^2 \int_0^t (t-s)^{-2\beta} ds \\
 &= \frac{\iota_\beta^2}{1-2\beta} |G|_{\mathcal{B}([0,t];E)}^2 t^{1-2\beta} < \infty,
 \end{aligned}$$

here we used the property (4.6).

To obtain the assertion in **Step 4**, it suffices to prove that  $I_2$  is  $\beta$ -Hölder continuous on  $[0, T]$ . Indeed, let  $0 \leq s < t \leq T$ . Using (4.6) and (4.7), we have

$$\begin{aligned}
 &\mathbb{E}|I_2(t) - I_2(s)|^2 \\
 &= \mathbb{E} \left| \int_s^t S(t-r)G(r)dw_r \right|^2 + \mathbb{E} \left| \int_0^s [S(t-r) - S(s-r)]G(r)dw_r \right|^2 \\
 &\leq c(E) \left[ \int_s^t |S(t-r)|^2 |G(r)|^2 dr + \int_0^s |S(t-r) - S(s-r)|^2 |G(r)|^2 dr \right] \\
 &\leq c(E) |G|_{\mathcal{B}([0,T];E)}^2 \left[ \iota_0^2 (t-s) + \int_0^s \left| \int_{s-r}^{t-r} AS(u)du \right|^2 dr \right]
 \end{aligned}$$



$$\leq c(E)|G|_{\mathcal{B}([0,T];E)}^2 \left[ \iota_0^2(t-s) + \iota_1^2 \int_0^s \left( \int_{s-r}^{t-r} u^{-1} du \right)^2 dr \right].$$

Fix  $\alpha \in (0, \frac{1}{2})$ . Dividing  $-1$  as  $-1 = -\alpha + \alpha - 1$ , we have

$$\left( \int_{s-r}^{t-r} u^{-1} du \right)^2 \leq (s-r)^{-2\alpha} \frac{(t-s)^{2\alpha}}{\alpha^2}, \quad (\text{see (4.30)}).$$

Therefore,

$$\begin{aligned} & \mathbb{E}|I_2(t) - I_2(s)|^2 \\ & \leq c(E)|G|_{\mathcal{B}([0,T];E)}^2 \left[ \iota_0^2(t-s) + \frac{\iota_1^2}{\alpha^2} (t-s)^{2\alpha} \int_0^s (s-r)^{-2\alpha} dr \right] \\ & \leq c(E)|G|_{\mathcal{B}([0,T];E)}^2 \left[ \iota_0^2(t-s) + \frac{\iota_1^2 s^{1-2\alpha}}{\alpha^2(1-2\alpha)} (t-s)^{2\alpha} \right] \\ & = c(E)|G|_{\mathcal{B}([0,T];E)}^2 \left[ \iota_0^2(t-s)^{1-2\alpha} + \frac{\iota_1^2 s^{1-2\alpha}}{\alpha^2(1-2\alpha)} \right] (t-s)^{2\alpha} \\ & \leq c(E)|G|_{\mathcal{B}([0,T];E)}^2 \left[ \iota_0^2 + \frac{\iota_1^2}{\alpha^2(1-2\alpha)} \right] T^{1-2\alpha} (t-s)^{2\alpha}. \end{aligned}$$

Applying Theorem 2.10 for the Gaussian process  $I_2(t)$ , we observe that  $I_2 \in \mathcal{C}^\alpha([0, T]; E)$  for any  $\alpha \in (0, \frac{1}{2})$ . In particular,  $I_2 \in \mathcal{C}^\beta([0, T]; E)$ .

Let us finally observe (4.24). Obviously, the estimate (4.32) still holds true, since it depends on neither (4.4) nor (4.5).

Meanwhile, using (4.8), we have

$$\begin{aligned} |I_2(t)|^2 & \leq c(E) \int_0^t |S(t-s)G(s)|^2 ds \\ & \leq c(E)\iota_0^2 |G|_{\mathcal{B}([0,t];E)}^2 \int_0^t e^{-2\nu(t-s)} ds \\ (4.34) \quad & = \frac{c(E)\iota_0^2 |G|_{\mathcal{B}([0,t];E)}^2}{2\nu} (1 - e^{-2\nu t}), \quad t \in [0, T]. \end{aligned}$$

Combining (4.32) and (4.34), we obtain that

$$\mathbb{E}|X(t)|^2 \leq 2c_1[\mathbb{E}|\xi|^2 + |F|_{\mathcal{F}^{\beta,\sigma}}^2] + \frac{c(E)\iota_0^2 |G|_{\mathcal{B}([0,t];E)}^2}{\nu} (1 - e^{-2\nu t}), \quad t \in [0, T].$$

Thus, (4.24) has been verified. It completes the proof.  $\square$

The following corollary is a direct consequence of Theorems 4.2-4.4.

COROLLARY 4.5. *Assume that (4.11) holds true and that  $\xi \in \mathcal{D}(A^\beta)$  a.s. Then there exists a strict solution  $X$  of (4.1) in the space:*

$$X \in \mathcal{C}((0, T]; \mathcal{D}(A)) \cap \mathcal{C}([0, T]; \mathcal{D}(A^\beta)) \cap \mathcal{C}^\beta([0, T]; E).$$

Furthermore,  $X$  satisfies the estimate (4.23) when (4.4) takes place, and satisfies the estimate (4.24) when (4.5) takes place.

**5. Semilinear evolution equations with additive noise.** In this section, we will study semilinear evolution equations with additive noise: the function  $F(t, x)$  is divided into two parts: one depends only on  $x$  and the other depends only on  $t$ , i.e.  $F(t, x) = F_1(X) + F_2(t)$ , and the function  $G(t, x) = G(t)$  depends only on  $t$ . Let us rewrite (1.1) into the form of semilinear evolution equations with additive noise.

$$(5.1) \quad \begin{cases} dX + AXdt = [F_1(X) + F_2(t)]dt + G(t)dw_t, & t \in (0, T], \\ X(0) = \xi, \end{cases}$$

where  $F_1$  is measurable from  $(\Omega_T \times E, \mathcal{P}_T \times \mathcal{B}(E))$  to  $(E, \mathcal{B}(E))$ ,  $F_2$  and  $G$  are non-random measurable functions from  $[0, T]$  to  $E$ .

Let fix constants  $\eta, \beta, \sigma$  such that

$$\begin{cases} 0 < \eta < \frac{1}{2}, \\ 0 \vee (2\eta - \frac{1}{2}) < \beta < \eta, \\ 0 < \sigma < \beta. \end{cases}$$

We suppose further that

(H1)  $F_1$  defines on the domain  $\mathcal{D}(A^\eta)$  and satisfies a Lipschitz condition of the form

$$(5.2) \quad |F_1(x) - F_1(y)| \leq c_{F_1} [|A^\eta(x - y)| + |A^{\tilde{\beta}}(x - y)|], \quad x, y \in \mathcal{D}(A^\eta)$$

with  $\tilde{\beta} = \beta$ , where  $c_{F_1}$  is some positive constant.

(H2)  $F_2 \in \mathcal{F}^{\beta, \sigma}((0, T]; E)$ .

(H3)  $G \in \mathcal{F}^{\beta + \frac{1}{2}, \sigma}((0, T]; E)$ .

### 5.1. Existence of local mild solutions.

THEOREM 5.1. *Let (4.2), (4.3), (H1), (H2) and (H3) be satisfied. Let  $\xi \in \mathcal{D}(A^\beta)$  such that  $\mathbb{E}|A^\beta \xi|^2 < \infty$ . Then (5.1) possesses a unique local mild solution  $X$  in the function space:*

$$(5.3) \quad X \in \mathcal{C}((0, T_{F_1, F_2, G, \xi}]; \mathcal{D}(A^\eta)) \cap \mathcal{C}([0, T_{F_1, F_2, G, \xi}]; \mathcal{D}(A^\beta)),$$

where  $T_{F_1, F_2, G, \xi}$  depends only on the squared norms  $|F_2|_{\mathcal{F}^{\beta, \sigma}}^2, |G|_{\mathcal{F}^{\beta + \frac{1}{2}, \sigma}}^2$  and  $\mathbb{E}|F_1(0)|^2$  and  $\mathbb{E}|A^\beta \xi|^2$ . Furthermore,  $X$  satisfies the estimates

$$(5.4) \quad \mathbb{E}|X(t)|^2 + \mathbb{E}|A^\beta X(t)|^2 \leq C_{F_1, F_2, G, \xi}, \quad t \in [0, T_{F_1, F_2, G, \xi}],$$

and

$$(5.5) \quad \mathbb{E}|A^\eta X(t)|^2 \leq C_{F_1, F_2, G, \xi} \left[ t^{-2(\eta-\beta)} + t^{2(1+\beta-2\eta)} + t^{2(1-\eta)} \right], \quad t \in (0, T_{F_1, F_2, G, \xi}]$$

with some constant  $C_{F_1, F_2, G, \xi}$  depending only on  $|F_2|_{\mathcal{F}^{\beta, \sigma}}^2, |G|_{\mathcal{F}^{\beta + \frac{1}{2}, \sigma}}^2, \mathbb{E}|F_1(0)|^2$  and  $\mathbb{E}|A^\beta \xi|^2$ .

PROOF. We shall use the fixed point theorem for contractions to prove existence and uniqueness of a local solution. For each  $S \in (0, T]$ , we set the underlying space:

$$\Xi(S) = \{Y \in \mathcal{C}((0, S]; \mathcal{D}(A^\eta)) \cap \mathcal{C}([0, S]; \mathcal{D}(A^\beta)) \text{ such that} \\ \sup_{0 < t \leq S} t^{2(\eta-\beta)} \mathbb{E}|A^\eta Y(t)|^2 + \sup_{0 \leq t \leq S} \mathbb{E}|A^\beta Y(t)|^2 < \infty\}.$$

Then up to indistinguishability,  $\Xi(S)$  is a Banach space with norm

$$(5.6) \quad |Y|_{\Xi(S)} = \left[ \sup_{0 < t \leq S} t^{2(\eta-\beta)} \mathbb{E}|A^\eta Y(t)|^2 + \sup_{0 \leq t \leq S} \mathbb{E}|A^\beta Y(t)|^2 \right]^{\frac{1}{2}}.$$

Let fix a constant  $\kappa > 0$  such that

$$(5.7) \quad \frac{\kappa^2}{2} > C_1 \vee C_2,$$

where two constants  $C_1$  and  $C_2$  will be fixed below. Consider a subset  $\Upsilon(S)$  of  $\Xi(S)$  which consists of all function  $Y \in \Xi(S)$  such that

$$(5.8) \quad \max \left\{ \sup_{0 < t \leq S} t^{2(\eta-\beta)} \mathbb{E}|A^\eta Y(t)|^2, \sup_{0 \leq t \leq S} \mathbb{E}|A^\beta Y(t)|^2 \right\} \leq \kappa^2.$$

Obviously,  $\Upsilon(S)$  is a nonempty closed subset of  $\Xi(S)$ .

For  $Y \in \Upsilon(S)$ , we define a function on  $[0, S]$

$$(5.9) \quad \Phi Y(t) = S(t)\xi + \int_0^t S(t-s)[F_1(Y(s)) + F_2(s)]ds + \int_0^t S(t-s)G(s)dw_s.$$

Our goal is then to verify that  $\Phi$  is a contraction mapping from  $\Upsilon(S)$  into itself, provided that  $S$  is sufficiently small, and that the fixed point of  $\Phi$  is

the desired solution of (5.1). For this purpose, we divide the proof into some steps.

**Step 1.** Let us show that  $\Phi Y \in \Upsilon(S)$  for every  $Y \in \Upsilon(S)$ . Let  $Y \in \Xi(S)$ . Due to (5.2) and (5.8), we observe that

$$\begin{aligned} \mathbb{E}|F_1(Y(t))|^2 &\leq \mathbb{E}[c_{F_1}|A^\eta Y(t)| + c_{F_1}|A^\beta Y(t)| + |F_1(0)|]^2 \\ (5.10) \quad &\leq 3[c_{F_1}^2 \mathbb{E}|A^\eta Y(t)|^2 + c_{F_1}^2 \mathbb{E}|A^\beta Y(t)|^2 + \mathbb{E}|F_1(0)|^2] \\ (5.11) \quad &\leq 3[c_{F_1}^2 \kappa^2 t^{2(\beta-\eta)} + c_{F_1}^2 \kappa^2 + \mathbb{E}|F_1(0)|^2], \quad t \in (0, S]. \end{aligned}$$

First, we shall show that  $\Phi Y$  satisfies (5.8). For  $\theta \in [\beta, \frac{1}{2}]$ , from (5.8) and (5.9), we have

$$\begin{aligned} &t^{2(\theta-\beta)} \mathbb{E}|A^\theta \{\Phi Y\}(t)|^2 \\ &\leq 3t^{2(\theta-\beta)} \mathbb{E} \left[ |A^\theta S(t)\xi|^2 + \left| \int_0^t A^\theta S(t-s)[F_1(Y(s)) + F_2(s)]ds \right|^2 \right. \\ &\quad \left. + \left| \int_0^t A^\theta S(t-s)G(s)dw_s \right|^2 \right] \\ &\leq 3t^{2(\theta-\beta)} |A^{\theta-\beta} S(t)|^2 \mathbb{E}|A^\beta \xi|^2 \\ &\quad + 6t^{2(\theta-\beta)} \mathbb{E} \left| \int_0^t A^\theta S(t-s)F_1(Y(s))ds \right|^2 \\ &\quad + 6t^{2(\theta-\beta)} \mathbb{E} \left| \int_0^t A^\theta S(t-s)F_2(s)ds \right|^2 \\ &\quad + 3c(E)t^{2(\theta-\beta)} \int_0^t |A^\theta S(t-s)G(s)|^2 ds. \end{aligned}$$

On the account of (2.5) and (4.6), we observe that

$$\begin{aligned} &t^{2(\theta-\beta)} \mathbb{E}|A^\theta \{\Phi Y\}(t)|^2 \\ &\leq 3\iota_{\theta-\beta}^2 \mathbb{E}|A^\beta \xi|^2 + 6t^{2(\theta-\beta)} \iota_\theta^2 \mathbb{E} \left[ \int_0^t (t-s)^{-\theta} |F_1(Y(s))| ds \right]^2 \\ &\quad + 6t^{2(\theta-\beta)} \iota_\theta^2 |F_2|_{\mathcal{F}^{\beta,\sigma}}^2 \left[ \int_0^t (t-s)^{-\theta} s^{\beta-1} ds \right]^2 \\ &\quad + 3c(E) \iota_\theta^2 |G|_{\mathcal{F}^{\beta+\frac{1}{2},\sigma}}^2 t^{2(\theta-\beta)} \int_0^t (t-s)^{-2\theta} s^{2\beta-1} ds \\ &\leq 3\iota_{\theta-\beta}^2 \mathbb{E}|A^\beta \xi|^2 + 6t^{1+2(\theta-\beta)} \iota_\theta^2 \int_0^t (t-s)^{-2\theta} \mathbb{E}|F_1(Y(s))|^2 ds \\ &\quad + 6\iota_\theta^2 |F_2|_{\mathcal{F}^{\beta,\sigma}}^2 \mathbf{B}(\beta, 1-\theta)^2 + 3c(E) \iota_\theta^2 |G|_{\mathcal{F}^{\beta+\frac{1}{2},\sigma}}^2 \mathbf{B}(2\beta, 1-2\theta). \end{aligned}$$

In view of (5.11), we obtain that

$$\begin{aligned}
& t^{2(\theta-\beta)} \mathbb{E}|A^\theta\{\Phi Y\}(t)|^2 \\
& \leq 3\iota_{\theta-\beta}^2 \mathbb{E}|A^\beta \xi|^2 \\
& \quad + 18t^{1+2(\theta-\beta)} \iota_\theta^2 \int_0^t (t-s)^{-2\theta} [c_{F_1}^2 \kappa^2 s^{2(\beta-\eta)} + c_{F_1}^2 \kappa^2 + \mathbb{E}|F_1(0)|^2] ds \\
& \quad + 6\iota_\theta^2 |F_2|_{\mathcal{F}^{\beta,\sigma}}^2 \mathbf{B}(\beta, 1-\theta)^2 + 3c(E) \iota_\theta^2 |G|_{\mathcal{F}^{\beta+\frac{1}{2},\sigma}}^2 \mathbf{B}(2\beta, 1-2\theta) \\
& = 3\iota_{\theta-\beta}^2 \mathbb{E}|A^\beta \xi|^2 + 18\iota_\theta^2 c_{F_1}^2 \kappa^2 t^{1+2(\theta-\eta)} \int_0^t (t-s)^{-2\theta} s^{2(\beta-\eta)} ds \\
& \quad + \frac{18\iota_\theta^2 [c_{F_1}^2 \kappa^2 + \mathbb{E}|F_1(0)|^2]}{1-2\theta} t^{2(1-\beta)} \\
& \quad + 6\iota_\theta^2 |F_2|_{\mathcal{F}^{\beta,\sigma}}^2 \mathbf{B}(\beta, 1-\theta)^2 + 3c(E) \iota_\theta^2 |G|_{\mathcal{F}^{\beta+\frac{1}{2},\sigma}}^2 \mathbf{B}(2\beta, 1-2\theta) \\
(5.12) \quad & = 3\iota_{\theta-\beta}^2 \mathbb{E}|A^\beta \xi|^2 + 6\iota_\theta^2 |F_2|_{\mathcal{F}^{\beta,\sigma}}^2 \mathbf{B}(\beta, 1-\theta)^2 \\
& \quad + 3c(E) \iota_\theta^2 |G|_{\mathcal{F}^{\beta+\frac{1}{2},\sigma}}^2 \mathbf{B}(2\beta, 1-2\theta) \\
& \quad + 18\iota_\theta^2 c_{F_1}^2 \kappa^2 \mathbf{B}(1+2\beta-2\eta, 1-2\theta) t^{2(1+\beta-2\eta)} \\
& \quad + \frac{18\iota_\theta^2 [c_{F_1}^2 \kappa^2 + \mathbb{E}|F_1(0)|^2]}{1-2\theta} t^{2(1-\beta)}.
\end{aligned}$$

We apply these estimates with  $\theta = \eta$  and  $\theta = \beta$ . Then it is observed that if  $C_1$  and  $C_2$  are fixed in such a way that

$$\begin{aligned}
C_1 & > 3\iota_{\eta-\beta}^2 \mathbb{E}|A^\beta \xi|^2 + 6\iota_\eta^2 |F_2|_{\mathcal{F}^{\beta,\sigma}}^2 \mathbf{B}(\beta, 1-\eta)^2 \\
& \quad + 3c(E) \iota_\eta^2 |G|_{\mathcal{F}^{\beta+\frac{1}{2},\sigma}}^2 \mathbf{B}(2\beta, 1-2\eta), \\
(5.13) \quad C_2 & > 3\iota_0^2 \mathbb{E}|A^\beta \xi|^2 + 6\iota_\beta^2 |F_2|_{\mathcal{F}^{\beta,\sigma}}^2 \mathbf{B}(\beta, 1-\beta)^2 \\
& \quad + 3c(E) \iota_\beta^2 |G|_{\mathcal{F}^{\beta+\frac{1}{2},\sigma}}^2 \mathbf{B}(2\beta, 1-2\beta),
\end{aligned}$$

and if  $S$  is sufficiently small, we have

$$\begin{aligned}
& t^{2(\eta-\beta)} \mathbb{E}|A^\eta\{\Phi Y\}(t)|^2 \leq C_1 + 18\iota_\eta^2 c_{F_1}^2 \kappa^2 \mathbf{B}(1+2\beta-2\eta, 1-2\eta) t^{2(1+\beta-2\eta)} \\
& \quad + \frac{18\iota_\eta^2 [c_{F_1}^2 \kappa^2 + \mathbb{E}|F_1(0)|^2]}{1-2\eta} t^{2(1-\beta)} \\
& \leq \frac{\kappa^2}{2} + 18\iota_\eta^2 c_{F_1}^2 \kappa^2 \mathbf{B}(1+2\beta-2\eta, 1-2\eta) t^{2(1+\beta-2\eta)} \\
& \quad + \frac{18\iota_\eta^2 [c_{F_1}^2 \kappa^2 + \mathbb{E}|F_1(0)|^2]}{1-2\eta} t^{2(1-\beta)} \\
(5.14) \quad & < \kappa^2, \quad t \in (0, S],
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}|A^\beta\{\Phi Y\}(t)|^2 &\leq C_2 + 18\iota_\beta^2 c_{F_1}^2 \kappa^2 \mathbf{B}(1 + 2\beta - 2\eta, 1 - 2\beta) t^{2(1+\beta-2\eta)} \\
&\quad + \frac{18\iota_\beta^2 [c_{F_1}^2 \kappa^2 + \mathbb{E}|F_1(0)|^2]}{1 - 2\beta} t^{2(1-\beta)} \\
&\leq \frac{\kappa^2}{2} + 18\iota_\beta^2 c_{F_1}^2 \kappa^2 \mathbf{B}(1 + 2\beta - 2\eta, 1 - 2\beta) t^{2(1+\beta-2\eta)} \\
&\quad + \frac{18\iota_\beta^2 [c_{F_1}^2 \kappa^2 + \mathbb{E}|F_1(0)|^2]}{1 - 2\beta} t^{2(1-\beta)} \\
(5.15) \quad &< \kappa^2, \quad t \in (0, S].
\end{aligned}$$

We thus have shown that

$$\max \left\{ \sup_{0 < t \leq S} t^{2(\eta-\beta)} \mathbb{E}|A^\eta \Phi Y(t)|^2, \sup_{0 \leq t \leq S} \mathbb{E}|A^\beta \Phi Y(t)|^2 \right\} \leq \kappa^2.$$

This means that  $\Phi Y$  satisfies (5.8).

Now, we shall show that

$$\Phi(Y) \in \mathcal{C}((0, S]; \mathcal{D}(A^\eta)) \cap \mathcal{C}([0, S]; \mathcal{D}(A^\beta)).$$

We divide  $\Phi Y$  into two parts:  $\Phi Y(t) = \Psi Y(t) + I_2(t)$ , where

$$\Psi Y(t) = S(t)\xi + \int_0^t S(t-s)[F_1(Y(s)) + F_2(s)]ds,$$

and  $I_2(t)$  is defined by (4.13).

As seen in **Step 2** of Theorem 4.4,

$$\begin{cases} A^\beta I_2(t) = A^\beta \int_0^t S(t-s)G(s)dw_s = \int_0^t A^\beta S(t-s)G(s)dw_s, \\ A^\beta I_2 \in \mathcal{C}([0, S]; E). \end{cases}$$

Furthermore, by using (2.5) and (4.6), we have

$$\begin{aligned}
\int_0^t |A^\eta S(t-s)G(s)|^2 ds &\leq \iota_\eta^2 |G|_{\mathcal{F}^{\beta+\frac{1}{2},\sigma}}^2 \int_0^t (t-s)^{-2\eta} s^{2\beta-1} ds \\
&= \iota_\eta^2 |G|_{\mathcal{F}^{\beta+\frac{1}{2},\sigma}}^2 \mathbf{B}(2\beta, 1 - 2\eta) t^{2(\beta-\eta)} < \infty, \quad t \in (0, S].
\end{aligned}$$

By the definition of stochastic integrals,  $\int_0^t A^\eta S(t-s)G(s)dw_s$ ,  $t \in (0, S]$  is well-defined and belongs to the space  $\mathcal{C}((0, S]; E)$ . We thus have verified that

$$I_2 \in \mathcal{C}((0, S]; \mathcal{D}(A^\eta)) \cap \mathcal{C}([0, S]; \mathcal{D}(A^\beta)).$$

Therefore, to finish **Step 1**, it suffices to show that

$$\Psi Y \in \mathcal{C}((0, S]; \mathcal{D}(A^\eta)) \cap \mathcal{C}([0, S]; \mathcal{D}(A^\beta)).$$

For  $0 < s < t \leq S$ , by the semigroup property we observe that

$$\begin{aligned} \Psi Y(t) - \Psi Y(s) &= S(t-s)S(s)\xi + S(t-s) \int_0^s S(s-r)[F_1(Y(r)) + F_2(r)]dr \\ &\quad - \Psi Y(s) + \int_s^t S(t-r)[F_1(Y(r)) + F_2(r)]dr \\ &= [S(t-s) - I]\Psi Y(s) + \int_s^t S(t-r)[F_1(Y(r)) + F_2(r)]dr. \end{aligned}$$

Let  $\rho \in (\frac{1}{2}, 1 - \eta)$ . In view of (4.6) and (4.9), we have

$$\begin{aligned} &|A^\eta[\Psi Y(t) - \Psi Y(s)]| \\ &\leq |[S(t-s) - I]A^{-\rho}||A^{\eta+\rho}\Psi Y(s)| + \int_s^t |A^\eta S(t-r)|[|F_1(Y(r))| + |F_2(r)|]dr \\ &\leq \frac{\iota_{1-\rho}(t-s)^\rho}{\rho} \left| A^{\eta+\rho} \left[ S(s)\xi + \int_0^s S(s-r)[F_1(Y(r)) + F_2(r)]dr \right] \right| \\ &\quad + \iota_\eta \int_s^t (t-r)^{-\eta} [|F_1(Y(r))| + |F_2(r)|]dr \\ &\leq \frac{\iota_{1-\rho}(t-s)^\rho}{\rho} |A^{\eta+\rho-\beta} S(s)||A^\beta \xi| \\ &\quad + \frac{\iota_{1-\rho}(t-s)^\rho}{\rho} \int_0^s |A^{\eta+\rho} S(s-r)||F_1(Y(r))|dr \\ &\quad + \frac{\iota_{1-\rho}(t-s)^\rho}{\rho} \int_0^s |A^{\eta+\rho} S(s-r)||F_2(r)|dr \\ &\quad + \iota_\eta \int_s^t (t-r)^{-\eta} |F_1(Y(r))|dr + \iota_\eta \int_s^t (t-r)^{-\eta} |F_2(r)|dr \\ &\leq \frac{\iota_{1-\rho}\iota_{\eta+\rho-\beta}(t-s)^\rho}{\rho} s^{-\eta-\rho+\beta} |A^\beta \xi| \\ &\quad + \frac{\iota_{1-\rho}\iota_{\eta+\rho}(t-s)^\rho}{\rho} \int_0^s (s-r)^{-\eta-\rho} |F_1(Y(r))|dr \\ &\quad + \frac{\iota_{1-\rho}\iota_{\eta+\rho}|F_2|_{\mathcal{F}^{\beta,\sigma}}(t-s)^\rho}{\rho} \int_0^s (s-r)^{-\eta-\rho} r^{\beta-1} dr \\ &\quad + \iota_\eta \int_s^t (t-r)^{-\eta} |F_1(Y(r))|dr \\ &\quad + \iota_\eta |F_2|_{\mathcal{F}^{\beta,\sigma}} \int_s^t (t-r)^{-\eta} r^{\beta-1} dr \end{aligned}$$

$$\begin{aligned}
&= \frac{\iota_{1-\rho}\iota_{\eta+\rho-\beta}}{\rho} |A^\beta \xi| s^{\beta-\eta-\rho} (t-s)^\rho \\
&\quad + \frac{\iota_{1-\rho}\iota_{\eta+\rho} |F_2|_{\mathcal{F}^{\beta,\sigma}} \mathbf{B}(\beta, 1-\eta-\rho)}{\rho} s^{\beta-\eta-\rho} (t-s)^\rho \\
&\quad + \iota_\eta |F_2|_{\mathcal{F}^{\beta,\sigma}} \int_s^t (t-r)^{-\eta} r^{\beta-1} dr \\
&\quad + \frac{\iota_{1-\rho}\iota_{\eta+\rho} (t-s)^\rho}{\rho} \int_0^s (s-r)^{-\eta-\rho} |F_1(Y(r))| dr \\
&\quad + \iota_\eta \int_s^t (t-r)^{-\eta} |F_1(Y(r))| dr.
\end{aligned}$$

Dividing  $\beta - 1$  as  $\beta - 1 = (\eta + \rho - 1) + (\beta - \eta - \rho)$ , we have

$$\begin{aligned}
\int_s^t (t-r)^{-\eta} r^{\beta-1} dr &\leq \int_s^t (t-r)^{-\eta} (r-s)^{\eta+\rho-1} s^{\beta-\eta-\rho} dr \\
&= \mathbf{B}(\eta + \rho, 1 - \eta) s^{\beta-\eta-\rho} (t-s)^\rho.
\end{aligned}$$

Hence,

$$\begin{aligned}
&|A^\eta[\Psi Y(t) - \Psi Y(s)]| \\
&\leq \frac{\iota_{1-\rho}\iota_{\eta+\rho-\beta}}{\rho} |A^\beta \xi| s^{\beta-\eta-\rho} (t-s)^\rho \\
&\quad + \left[ \frac{\iota_{1-\rho}\iota_{\eta+\rho} \mathbf{B}(\beta, 1-\eta-\rho)}{\rho} + \iota_\eta \mathbf{B}(\eta + \rho, 1 - \eta) \right] |F_2|_{\mathcal{F}^{\beta,\sigma}} s^{\beta-\eta-\rho} (t-s)^\rho \\
&\quad + \frac{\iota_{1-\rho}\iota_{\eta+\rho} (t-s)^\rho}{\rho} \int_0^s (s-r)^{-\eta-\rho} |F_1(Y(r))| dr \\
&\quad + \iota_\eta \int_s^t (t-r)^{-\eta} |F_1(Y(r))| dr.
\end{aligned}$$

Taking expectation of the square of the both hand sides of the above inequality, we see that

$$\begin{aligned}
&\mathbb{E}|A^\eta[\Psi Y(t) - \Psi Y(s)]|^2 \\
&\leq \frac{4\iota_{1-\rho}^2\iota_{\eta+\rho-\beta}^2}{\rho^2} \mathbb{E}|A^\beta \xi|^2 s^{2(\beta-\eta-\rho)} (t-s)^{2\rho} \\
&\quad + 4 \left[ \frac{\iota_{1-\rho}\iota_{\eta+\rho} \mathbf{B}(\beta, 1-\eta-\rho)}{\rho} + \iota_\eta \mathbf{B}(\eta + \rho, 1 - \eta) \right]^2 \\
&\quad \times |F_2|_{\mathcal{F}^{\beta,\sigma}}^2 s^{2(\beta-\eta-\rho)} (t-s)^{2\rho} \\
&\quad + \frac{4\iota_{1-\rho}^2\iota_{\eta+\rho}^2}{\rho^2} (t-s)^{2\rho} \mathbb{E} \left[ \int_0^s (s-r)^{-\eta-\rho} |F_1(Y(r))| dr \right]^2
\end{aligned}$$



$$+ 4\iota_\eta^2 \mathbb{E} \left[ \int_s^t (t-r)^{-\eta} |F_1(Y(r))| dr \right]^2.$$

Since

$$\begin{aligned} & \left[ \int_0^s (s-r)^{-\eta-\rho} |F_1(Y(r))| dr \right]^2 \\ &= \left[ \int_0^s (s-r)^{-\frac{\eta-\rho}{2}} (s-r)^{-\frac{\eta-\rho}{2}} |F_1(Y(r))| dr \right]^2 \\ &\leq \int_0^s (s-r)^{-\eta-\rho} dr \int_0^s (s-r)^{-\eta-\rho} |F_1(Y(r))|^2 dr \\ &= \frac{s^{1-\eta-\rho}}{1-\eta-\rho} \int_0^s (s-r)^{-\eta-\rho} |F_1(Y(r))|^2 dr, \end{aligned}$$

we have

$$\begin{aligned} (5.16) \quad & \mathbb{E} |A^\eta [\Psi Y(t) - \Psi Y(s)]|^2 \\ &\leq \frac{4\iota_{1-\rho}^2 \iota_{\eta+\rho-\beta}^2}{\rho^2} \mathbb{E} |A^\beta \xi|^2 s^{2(\beta-\eta-\rho)} (t-s)^{2\rho} \\ &\quad + 4 \left[ \frac{\iota_{1-\rho} \iota_{\eta+\rho} \mathbf{B}(\beta, 1-\eta-\rho)}{\rho} + \iota_\eta \mathbf{B}(\eta+\rho, 1-\eta) \right]^2 \\ &\quad \times |F_2|_{\mathcal{F}^{\beta,\sigma}}^2 s^{2(\beta-\eta-\rho)} (t-s)^{2\rho} \\ &\quad + \frac{4\iota_{1-\rho}^2 \iota_{\eta+\rho}^2}{\rho^2 (1-\eta-\rho)} (t-s)^{2\rho} s^{1-\eta-\rho} \int_0^s (s-r)^{-\eta-\rho} \mathbb{E} |F_1(Y(r))|^2 dr \\ &\quad + 4\iota_\eta^2 (t-s) \int_s^t (t-r)^{-2\eta} \mathbb{E} |F_1(Y(r))|^2 dr. \end{aligned}$$

Using (5.11), we have

$$\begin{aligned} & \int_0^s (s-r)^{-\eta-\rho} \mathbb{E} |F_1(Y(r))|^2 dr \\ &\leq 3c_{F_1}^2 \kappa^2 \int_0^s (s-r)^{-\eta-\rho} r^{2(\beta-\eta)} dr + 3[c_{F_1}^2 \kappa^2 + \mathbb{E} |F_1(0)|^2] \int_0^s (s-r)^{-\eta-\rho} dr \\ (5.17) \quad &= 3c_{F_1}^2 \kappa^2 \mathbf{B}(1+2\beta-2\eta, 1-\eta-\rho) s^{1+2\beta-3\eta-\rho} \\ &\quad + \frac{3[c_{F_1}^2 \kappa^2 + \mathbb{E} |F_1(0)|^2] s^{1-\eta-\rho}}{1-\eta-\rho}, \end{aligned}$$

and

$$\int_s^t (t-r)^{-2\eta} \mathbb{E} |F_1(Y(r))|^2 dr$$

$$\begin{aligned}
&\leq 3 \int_s^t (t-r)^{-2\eta} [c_{F_1}^2 \kappa^2 r^{2(\beta-\eta)} + c_{F_1}^2 \kappa^2 + \mathbb{E}|F_1(0)|^2] dr \\
(5.18) \quad &= 3c_{F_1}^2 \kappa^2 \int_s^t (t-r)^{-2\eta} r^{2(\beta-\eta)} dr + \frac{3[c_{F_1}^2 \kappa^2 + \mathbb{E}|F_1(0)|^2]}{1-2\eta} (t-s)^{1-2\eta}.
\end{aligned}$$

Dividing  $2(\beta-\eta)$  as  $2(\beta-\eta) = (\beta-\frac{1}{2}) + (\frac{1}{2} + \beta - 2\eta)$ , we have

$$\begin{aligned}
&\int_s^t (t-r)^{-2\eta} r^{2(\beta-\eta)} dr \leq \int_s^t (t-r)^{-2\eta} (r-s)^{\beta-\frac{1}{2}} t^{\frac{1}{2}+\beta-2\eta} dr \\
(5.19) \quad &= \mathbf{B}(\frac{1}{2} + \beta, 1-2\eta) t^{\frac{1}{2}+\beta-2\eta} (t-s)^{\frac{1}{2}+\beta-2\eta}.
\end{aligned}$$

By virtue of (5.17), (5.18) and (5.19), it follows from (5.16) that

$$\begin{aligned}
&\mathbb{E}|A^\eta[\Psi Y(t) - \Psi Y(s)]|^2 \\
&\leq \frac{4\iota_{1-\rho}^2 \iota_{\eta+\rho-\beta}^2}{\rho^2} \mathbb{E}|A^\beta \xi|^2 s^{2(\beta-\eta-\rho)} (t-s)^{2\rho} \\
&\quad + 4 \left[ \frac{\iota_{1-\rho} \iota_{\eta+\rho} \mathbf{B}(\beta, 1-\eta-\rho)}{\rho} + \iota_\eta \mathbf{B}(\eta+\rho, 1-\eta) \right]^2 \\
&\quad \times |F_2|_{\mathcal{F}^{\beta,\sigma}}^2 s^{2(\beta-\eta-\rho)} (t-s)^{2\rho} \\
&\quad + \frac{12\iota_{1-\rho}^2 \iota_{\eta+\rho}^2 c_{F_1}^2 \kappa^2 \mathbf{B}(1+2\beta-2\eta, 1-\eta-\rho)}{\rho^2(1-\eta-\rho)} (t-s)^{2\rho} s^{2(1+\beta-2\eta-\rho)} \\
&\quad + \frac{12\iota_{1-\rho}^2 \iota_{\eta+\rho}^2 [c_{F_1}^2 \kappa^2 + \mathbb{E}|F_1(0)|^2]}{\rho^2(1-\eta-\rho)^2} (t-s)^{2\rho} s^{2(1-\eta-\rho)} \\
&\quad + 12\iota_\eta^2 c_{F_1}^2 \kappa^2 \mathbf{B}(\frac{1}{2} + \beta, 1-2\eta) t^{\frac{1}{2}+\beta-2\eta} (t-s)^{\frac{3}{2}+\beta-2\eta} \\
&\quad + \frac{12\iota_\eta^2 [c_{F_1}^2 \kappa^2 + \mathbb{E}|F_1(0)|^2]}{1-2\eta} (t-s)^{2(1-\eta)}.
\end{aligned}$$

Since this estimate holds for any  $\rho \in (\frac{1}{2}, 1-\eta)$ , and since  $1 < \frac{3}{2} + \beta - 2\eta < 2(1-\eta)$ , the Kolmogorov test then provides that  $A^\eta \Psi Y$  is Hölder continuous on  $(0, S]$  with an arbitrary exponent smaller than  $\frac{1+2\beta}{4} - \eta$ . As a consequence,  $\Psi Y \in \mathcal{C}((0, S]; \mathcal{D}(A^\eta))$ .

Similarly, we also have

$$\begin{aligned}
&\mathbb{E}|A^\beta[\Psi Y(t) - \Psi Y(s)]|^2 \\
&\leq \frac{4\iota_{1-\rho}^2 \iota_\rho^2}{\rho^2} \mathbb{E}|A^\beta \xi|^2 s^{-2\rho} (t-s)^{2\rho} \\
&\quad + 4 \left[ \frac{\iota_{1-\rho} \iota_{\beta+\rho} \mathbf{B}(\beta, 1-\beta-\rho)}{\rho} + \iota_\beta \mathbf{B}(\beta+\rho, 1-\beta) \right]^2
\end{aligned}$$

$$\begin{aligned}
& \times |F_2|_{\mathcal{F}^{\beta,\sigma}}^2 s^{-2\rho} (t-s)^{2\rho} \\
& + \frac{12\iota_{1-\rho}^2 \iota_{\beta+\rho}^2 c_{F_1}^2 \kappa^2 \mathbf{B}(1, 1-\beta-\rho)}{\rho^2(1-\beta-\rho)} (t-s)^{2\rho} s^{2(1-\eta-\rho)} \\
& + \frac{12\iota_{1-\rho}^2 \iota_{\beta+\rho}^2 [c_{F_1}^2 \kappa^2 + \mathbb{E}|F_1(0)|^2]}{\rho^2(1-\beta-\rho)^2} (t-s)^{2\rho} s^{2(1-\beta-\rho)} \\
& + 12\iota_\beta^2 c_{F_1}^2 \kappa^2 \mathbf{B}\left(\frac{1}{2} + \beta, 1-2\beta\right) t^{\frac{1}{2}-\beta} (t-s)^{\frac{3}{2}-\beta} \\
& + \frac{12\iota_\beta^2 [c_{F_1}^2 \kappa^2 + \mathbb{E}|F_1(0)|^2]}{1-2\beta} (t-s)^{2(1-\beta)}.
\end{aligned}$$

The Kolmogorov test again provides that  $\Psi Y \in \mathcal{C}((0, S]; \mathcal{D}(A^\beta))$ .

It remains to show that  $A^\beta \Psi Y$  is continuous at  $t = 0$ . Indeed, we already know by Theorem 4.4 that

$$A^\beta \left[ S(t)\xi + \int_0^t S(t-s)F_2(s)ds + \int_0^t S(t-s)G(s)dw_s \right]$$

is continuous at  $t = 0$ . Meanwhile, using (4.6) and (5.11), we have

$$\begin{aligned}
& \mathbb{E} \left| A^\beta \int_0^t S(t-s)F_1(Y(s))ds \right|^2 \\
& \leq \mathbb{E} \left[ \int_0^t |A^\beta S(t-s)| |F_1(Y(s))| ds \right]^2 \\
& \leq \iota_\beta^2 \mathbb{E} \left[ \int_0^t (t-s)^{-\beta} |F_1(Y(s))| ds \right]^2 \\
& \leq \iota_\beta^2 t \int_0^t (t-s)^{-2\beta} \mathbb{E}|F_1(Y(s))|^2 ds \\
& \leq 3\iota_\beta^2 t \int_0^t (t-s)^{-2\beta} [c_{F_1}^2 \kappa^2 s^{2(\beta-\eta)} + c_{F_1}^2 \kappa^2 + \mathbb{E}|F_1(0)|^2] ds \\
(5.20) \quad & = 3\iota_\beta^2 \left[ c_{F_1}^2 \kappa^2 \mathbf{B}(1+2\beta-2\eta, 1-2\beta) t^{2(1-\eta)} + \frac{c_{F_1}^2 \kappa^2 + \mathbb{E}|F_1(0)|^2}{1-2\beta} t^{2(1-\beta)} \right] \\
& \rightarrow 0 \quad \text{as } t \searrow 0.
\end{aligned}$$

Therefore, there exists a decreasing sequence  $\{t_n\}$  converging to 0 such that

$$\lim_{n \rightarrow \infty} A^\beta \int_0^{t_n} S(t_n-s)F_1(Y(s))ds = 0.$$

By the continuity of  $A^\beta \int_0^{t_n} S(t_n-s)F_1(Y(s))ds$  on  $(0, S]$ , we conclude that

$$\lim_{t \rightarrow 0} A^\beta \int_0^t S(t-s)F_1(Y(s))ds = 0,$$

i.e.  $A^\beta \int_0^t S(t-s)F_1(Y(s))ds$  is continuous at  $t = 0$ . We thus have shown that  $A^\beta \Psi Y$  is continuous at  $t = 0$ .

**Step 2.** Let us show that  $\Phi$  is a contraction mapping of  $\Xi(S)$ , provided  $S > 0$  is sufficiently small. Let  $Y_1, Y_2 \in \Xi(S)$  and  $\theta \in [0, \frac{1}{2})$ . From (5.9), we see that

$$\begin{aligned} & t^{2(\theta-\beta)} \mathbb{E} |A^\theta [\Phi Y_1(t) - \Phi Y_2(t)]|^2 \\ &= t^{2(\theta-\beta)} \mathbb{E} \left| \int_0^t A^\theta S(t-s) [F_1(Y_1(s)) - F_1(Y_2(s))] ds \right|^2 \\ &\leq t^{2(\theta-\beta)} \mathbb{E} \left[ \int_0^t |A^\theta S(t-s)| |F_1(Y_1(s)) - F_1(Y_2(s))| ds \right]^2. \end{aligned}$$

By virtue of (4.6), (5.2) and (5.6), we have

$$\begin{aligned} & t^{2(\theta-\beta)} \mathbb{E} |A^\theta [\Phi Y_1(t) - \Phi Y_2(t)]|^2 \\ &\leq c_{F_1}^2 \iota_\theta^2 t^{2(\theta-\beta)} \\ &\quad \times \mathbb{E} \left[ \int_0^t (t-s)^{-\theta} \{ |A^\eta(Y_1(s) - Y_2(s))| + |A^\beta(Y_1(s) - Y_2(s))| \} ds \right]^2 \\ &\leq c_{F_1}^2 \iota_\theta^2 t^{1+2(\theta-\beta)} \\ &\quad \times \mathbb{E} \int_0^t (t-s)^{-2\theta} \{ |A^\eta(Y_1(s) - Y_2(s))| + |A^\beta(Y_1(s) - Y_2(s))| \}^2 ds \\ &\leq 2c_{F_1}^2 \iota_\theta^2 t^{1+2(\theta-\beta)} \int_0^t (t-s)^{-2\theta} \mathbb{E} |A^\eta(Y_1(s) - Y_2(s))|^2 ds \\ &\quad + 2c_{F_1}^2 \iota_\theta^2 t^{1+2(\theta-\beta)} \int_0^t (t-s)^{-2\theta} \mathbb{E} |A^\beta(Y_1(s) - Y_2(s))|^2 ds \\ &\leq 2c_{F_1}^2 \iota_\theta^2 t^{1+2(\theta-\beta)} \int_0^t (t-s)^{-2\theta} s^{2(\beta-\eta)} |Y_1 - Y_2|_{\Xi(S)}^2 ds \\ &\quad + 2c_{F_1}^2 \iota_\theta^2 t^{1+2(\theta-\beta)} \int_0^t (t-s)^{-2\theta} |Y_1 - Y_2|_{\Xi(S)}^2 ds \\ &= 2c_{F_1}^2 \iota_\theta^2 \left[ \mathbf{B}(1+2\beta-2\eta, 1-2\theta) t^{2(1-\eta)} + \frac{t^{2(1-\beta)}}{1-2\theta} \right] |Y_1 - Y_2|_{\Xi(S)}^2 \\ &\leq 2c_{F_1}^2 \left[ \iota_\theta^2 \mathbf{B}(1+2\beta-2\eta, 1-2\theta) + \frac{\iota_\theta^2 S^{2(\eta-\beta)}}{1-2\theta} \right] S^{2(1-\eta)} |Y_1 - Y_2|_{\Xi(S)}^2. \end{aligned}$$

Applying these estimates with  $\theta = \eta$  and  $\theta = \beta$ , we conclude that

$$\begin{aligned} & |\Phi Y_1 - \Phi Y_2|_{\Xi(S)}^2 \\ &= \left[ \sup_{0 < t \leq S} t^{2(\eta-\beta)} \mathbb{E} |A^\eta [\Phi Y_1(t) - \Phi Y_2(t)]|^2 + \sup_{0 \leq t \leq S} \mathbb{E} |A^\beta [\Phi Y_1(t) - \Phi Y_2(t)]|^2 \right]^{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned}
(5.21) \quad & \leq 2c_{F_1}^2 \left[ \iota_\eta^2 \mathbf{B}(1 + 2\beta - 2\eta, 1 - 2\eta) + \iota_\beta^2 \mathbf{B}(1 + 2\beta - 2\eta, 1 - 2\beta) \right. \\
& \quad \left. + \frac{\iota_\eta^2 S^{2(\eta-\beta)}}{1 - 2\eta} + \frac{\iota_\beta^2 S^{2(\eta-\beta)}}{1 - 2\beta} \right] S^{2(1-\eta)} |Y_1 - Y_2|_{\Xi(S)}^2.
\end{aligned}$$

This shows that  $\Phi$  is contraction in  $\Xi(S)$ , provided  $S > 0$  is sufficiently small.

**Step 3.** Let us show existence of a local mild solution. Let  $S > 0$  be sufficiently small in such a way that  $\Phi$  maps  $\Upsilon(S)$  into itself and is contraction with respect to the norm of  $\Xi(S)$ . Due to **Step 1** and **Step 2**,  $S = T_{F_1, F_2, G, \xi}$  can be determined by  $\mathbb{E}|F_1(0)|^2$ ,  $|F_2|_{\mathcal{F}^{\beta, \sigma}}^2$ ,  $|G|_{\mathcal{F}^{\beta+\frac{1}{2}, \sigma}}^2$  and  $\mathbb{E}|A^\beta \xi|^2$ . Thanks to the fixed point theorem, there exists a unique function  $X \in \Upsilon(T_{F_1, F_2, G, \xi})$  such that  $X = \Phi X$ . This means that  $X$  is a mild solution of (5.1) in the function space:

$$X \in \mathcal{C}((0, T_{F_1, F_2, G, \xi}]; \mathcal{D}(A^\eta)) \cap \mathcal{C}([0, T_{F_1, F_2, G, \xi}]; \mathcal{D}(A^\beta)).$$

**Step 4.** Let us verify the estimate (5.4). We have

$$\begin{aligned}
(5.22) \quad X(t) &= \left[ S(t)\xi + \int_0^t S(t-s)F_2(s)ds + \int_0^t S(t-s)G(s)dw_s \right] \\
&\quad + \int_0^t S(t-s)F_1(X(s))ds \\
&= X_1(t) + X_2(t), \quad t \in [0, T_{F_1, F_2, G, \xi}].
\end{aligned}$$

On the account of Theorem 4.4, we have an estimate for the first term

$$\mathbb{E}|X_1(t)|^2 \leq \rho_1 [\mathbb{E}|\xi|^2 + |F|_{\mathcal{F}^{\beta, \sigma}}^2 + |G|_{\mathcal{F}^{\beta+\frac{1}{2}, \sigma}}^2],$$

where  $\rho_1$  is a positive constant depending only on  $A, \beta$  and  $\sigma$ .

For the second term, in view of (4.7) and (5.11), we observe that

$$\begin{aligned}
\mathbb{E}|X_2(t)|^2 &\leq \mathbb{E} \left[ \int_0^t |S(t-s)| |F_1(X(s))| ds \right]^2 \\
&\leq t \int_0^t \iota_0^2 \mathbb{E}|F_1(X(s))|^2 ds \\
&\leq 3t \int_0^t \iota_0^2 [c_{F_1}^2 \kappa^2 s^{2(\beta-\eta)} + c_{F_1}^2 \kappa^2 + \mathbb{E}|F_1(0)|^2] ds \\
&= \frac{3c_{F_1}^2 \kappa^2 \iota_0^2}{1 + 2(\beta - \eta)} t^{2(1+\beta-\eta)} + 3[c_{F_1}^2 \kappa^2 + \mathbb{E}|F_1(0)|^2] \iota_0^2 t^2
\end{aligned}$$

for every  $t \in [0, T_{F_1, F_2, G, \xi}]$ . Hence,

$$\begin{aligned}
 \mathbb{E}|X(t)|^2 &\leq 2\mathbb{E}|X_1(t)|^2 + 2\mathbb{E}|X_2(t)|^2 \\
 &\leq 2\rho_1[\mathbb{E}|\xi|^2 + \mathbb{E}|F|_{\mathcal{F}^{\beta, \sigma}}^2 + |G|_{\mathcal{F}^{\beta+\frac{1}{2}, \sigma}}^2] \\
 &\quad + \frac{6c_{F_1}^2 \kappa^2 \iota_0^2}{1+2(\beta-\eta)} t^{2(1+\beta-\eta)} + 6[c_{F_1}^2 \kappa^2 + \mathbb{E}|F_1(0)|^2] \iota_0^2 t^2 \\
 (5.23) \quad &\leq 2\rho_1[\mathbb{E}|\xi|^2 + \mathbb{E}|F|_{\mathcal{F}^{\beta, \sigma}}^2 + |G|_{\mathcal{F}^{\beta+\frac{1}{2}, \sigma}}^2] \\
 &\quad + \frac{6c_{F_1}^2 \kappa^2 \iota_0^2}{1+2(\beta-\eta)} T^{2(1+\beta-\eta)} + 6[c_{F_1}^2 \kappa^2 + \mathbb{E}|F_1(0)|^2] \iota_0^2 T^2
 \end{aligned}$$

for every  $t \in [0, T_{F_1, F_2, G, \xi}]$ .

On the other hand, in virtue of (4.7), (4.25), (4.26), (5.20) and (5.22), for every  $t \in [0, T_{F_1, F_2, G, \xi}]$  we observe that

$$\begin{aligned}
 &\mathbb{E}|A^\beta X(t)|^2 \\
 &\leq 4\mathbb{E}|S(t)A^\beta \xi|^2 + 4\mathbb{E}\left|\int_0^t A^\beta S(t-s)F_2(s)ds\right|^2 \\
 &\quad + 4\mathbb{E}\left|\int_0^t A^\beta S(t-s)F_1(X(s))ds\right|^2 + 4\mathbb{E}\left|\int_0^t A^\beta S(t-s)G(s)dw_s\right|^2 \\
 &\leq 4\iota_0^2 \mathbb{E}|A^\beta \xi|^2 + 4\iota_\beta^2 |F_2|_{\mathcal{F}^{\beta, \sigma}}^2 \mathbf{B}(\beta, 1-\beta)^2 \\
 &\quad + 12\iota_\beta^2 \left[ c_{F_1}^2 \kappa^2 \mathbf{B}(1+2\beta-2\eta, 1-2\beta) t^{2(1-\eta)} + \frac{c_{F_1}^2 \kappa^2 + \mathbb{E}|F_1(0)|^2}{1-2\beta} t^{2(1-\beta)} \right] \\
 &\quad + 4c(E) \int_0^t |A^\beta S(t-s)G(s)|^2 ds \\
 &\leq 4\iota_0^2 \mathbb{E}|A^\beta \xi|^2 + 4\iota_\beta^2 |F_2|_{\mathcal{F}^{\beta, \sigma}}^2 \mathbf{B}(\beta, 1-\beta)^2 \\
 &\quad + 12\iota_\beta^2 \left[ c_{F_1}^2 \kappa^2 \mathbf{B}(1+2\beta-2\eta, 1-2\beta) t^{2(1-\eta)} + \frac{c_{F_1}^2 \kappa^2 + \mathbb{E}|F_1(0)|^2}{1-2\beta} t^{2(1-\beta)} \right] \\
 &\quad + 4c(E) \iota_\beta^2 |G|_{\mathcal{F}^{\beta+\frac{1}{2}, \sigma}}^2 \mathbf{B}(2\beta, 1-2\beta) \\
 (5.24) \quad &\leq 4\iota_0^2 \mathbb{E}|A^\beta \xi|^2 + 4\iota_\beta^2 |F_2|_{\mathcal{F}^{\beta, \sigma}}^2 \mathbf{B}(\beta, 1-\beta)^2 + 4c(E) \iota_\beta^2 |G|_{\mathcal{F}^{\beta+\frac{1}{2}, \sigma}}^2 \mathbf{B}(2\beta, 1-2\beta) \\
 &\quad + 12\iota_\beta^2 \left[ c_{F_1}^2 \kappa^2 \mathbf{B}(1+2\beta-2\eta, 1-2\beta) T^{2(1-\eta)} + \frac{c_{F_1}^2 \kappa^2 + \mathbb{E}|F_1(0)|^2}{1-2\beta} T^{2(1-\beta)} \right].
 \end{aligned}$$

Combining (5.23) and (5.24), we obtain (5.4).

**Step 5.** Let us verify the estimate (5.5). From (4.6) and (5.22), we observe that

$$\mathbb{E}|A^\eta X(t)|^2$$

$$\begin{aligned}
&\leq 4\mathbb{E}|A^\eta S(t)\xi|^2 + 4\left[\int_0^t |A^\eta S(t-s)||F_2(s)|ds\right]^2 \\
&\quad + 4\mathbb{E}\left[\int_0^t |A^\eta S(t-s)||F_1(X(s))|ds\right]^2 + 4c(E)\int_0^t |A^\eta S(t-s)G(s)|^2 ds \\
&= 4\mathbb{E}|A^{\eta-\beta}S(t)A^\beta\xi|^2 + 4J_1 + 4J_2 + 4J_3 \\
&\leq 4\iota_{\eta-\beta}^2\mathbb{E}|A^\beta\xi|^2 t^{-2(\eta-\beta)} + 4J_1 + 4J_2 + 4J_3, \quad t \in (0, T_{F_1, F_2, G, \xi}].
\end{aligned}$$

We shall give estimates for  $J_1, J_2$  and  $J_3$ . For  $J_1$ , similarly to (4.25), we have

$$J_1 \leq \iota_\eta^2 |F|_{\mathcal{F}^{\beta, \sigma}}^2 \mathbf{B}(\beta, 1 - \eta)^2 t^{2(\beta - \eta)}.$$

For  $J_2$ , similarly to (5.20), we conclude that

$$J_2 \leq 3\iota_\eta^2 \left[ c_{F_1}^2 \kappa^2 \mathbf{B}(1 + 2\beta - 2\eta, 1 - 2\eta) t^{2(1 + \beta - 2\eta)} + \frac{c_{F_1}^2 \kappa^2 + \mathbb{E}|F_1(0)|^2}{1 - 2\eta} t^{2(1 - \eta)} \right].$$

For  $J_3$ , similarly to (4.26), we obtain that

$$J_3 \leq c(E) \iota_\eta^2 |G|_{\mathcal{F}^{\beta + \frac{1}{2}, \sigma}}^2 \mathbf{B}(2\beta, 1 - 2\eta) t^{2(\beta - \eta)}.$$

Thus, (5.5) is verified.

**Step 6.** Let us finally show uniqueness of the local mild solution. Let  $\bar{X}$  be any other local mild solution to (5.1) on the interval  $[0, T_{F_1, F_2, G, \xi}]$  which belongs to the space  $\mathcal{C}((0, T_{F_1, F_2, G, \xi}]; \mathcal{D}(A^\eta)) \cap \mathcal{C}([0, T_{F_1, F_2, G, \xi}]; \mathcal{D}(A^{\frac{\beta}{2}}))$ .

The formula

$$\bar{X}(t) = S(t)\xi + \int_0^t S(t-s)[F_1(\bar{X}(s)) + F_2(s)]ds + \int_0^t S(t-s)G(s)dw_s$$

jointed with (5.22) yields that

$$X(t) - \bar{X}(t) = \int_0^t S(t-s)[F_1(X(s)) - F_1(\bar{X}(s))]ds, \quad t \in [0, T_{F_1, F_2, G, \xi}].$$

We can then repeat the same arguments as in **Step 2** to deduce that

(5.25)

$$\begin{aligned}
&|X - \bar{X}|_{\Xi(\bar{T})}^2 \\
&\leq 2c_{F_1}^2 \left[ \iota_\eta^2 \mathbf{B}(1 + 2\beta - 2\eta, 1 - 2\eta) + \iota_\beta^2 \mathbf{B}(1 + 2\beta - 2\eta, 1 - 2\beta) + \frac{\iota_\eta^2 \bar{T}^{2(\eta - \beta)}}{1 - 2\eta} \right. \\
&\quad \left. + \frac{\iota_\beta^2 \bar{T}^{2(\eta - \beta)}}{1 - 2\beta} \right] \bar{T}^{2(1 - \eta)} |X - \bar{X}|_{\Xi(\bar{T})}^2 \quad \text{for any } \bar{T} \in (0, T_{F_1, F_2, G, \xi}].
\end{aligned}$$

Let  $\bar{T}$  be a positive constant such that

$$2c_{F_1}^2 \left[ \iota_\eta^2 \mathbf{B}(1+2\beta-2\eta, 1-2\eta) + \iota_\beta^2 \mathbf{B}(1+2\beta-2\eta, 1-2\beta) \right. \\ \left. + \frac{\iota_\eta^2 \bar{T}^{2(\eta-\beta)}}{1-2\eta} + \frac{\iota_\beta^2 \bar{T}^{2(\eta-\beta)}}{1-2\beta} \right] \bar{T}^{2(1-\eta)} < 1.$$

It then follows from (5.25) that  $X(t) = \bar{X}(t)$  a.s. for every  $t \in [0, \bar{T}]$ . We repeat the same procedure with initial time  $\bar{T}$  and initial value  $X(\bar{T}) = \bar{X}(\bar{T})$  to derive that  $X(\bar{T}+t) = \bar{X}(\bar{T}+t)$  a.s. for every  $t \in [0, \bar{T}]$ . This means that  $X(t) = \bar{X}(t)$  a.s. on a larger interval  $[0, 2\bar{T}]$ . We continue this procedure by finite times, the extended interval can cover the given interval  $[0, T_{F_1, F_2, G, \xi}]$ . Therefore, for every  $t \in [0, T_{F_1, F_2, G, \xi}]$ ,  $X(t) = \bar{X}(t)$  a.s.  $\square$

**COROLLARY 5.2** (global existence). *Assume that in Theorem 5.1 the constant  $C_{F_1, F_2, G, \xi}$  is independent of  $T_{F_1, F_2, G, \xi}$  for every  $\xi \in \mathcal{D}(A^\beta)$  such that  $\mathbb{E}|A^\beta \xi|^2 < \infty$ . Then (5.1) possesses a unique mild solution on the interval  $[0, T]$ .*

**PROOF.** Let extend the functions  $F_2$  and  $G$  to functions  $\bar{F}_2$  and  $\bar{G}$  defined on the whole interval  $[0, \infty)$  by putting  $\bar{F}_2(t) \equiv F_2(T)$  and  $\bar{G}(t) \equiv G(T)$  for  $T < t < \infty$ . It is obvious that

$$|\bar{F}_2|_{\mathcal{F}^{\beta, \sigma}((a, b]; E)} \leq |F_2|_{\mathcal{F}^{\beta, \sigma}((a, b]; E)} \quad \text{and} \quad |\bar{G}|_{\mathcal{F}^{\beta + \frac{1}{2}, \sigma}((a, b]; E)} \leq |G|_{\mathcal{F}^{\beta + \frac{1}{2}, \sigma}((a, b]; E)}$$

for any interval  $(a, b] \subset [0, \infty)$ .

Let  $\bar{\xi} = X(\frac{T_{F_1, F_2, G, \xi}}{2})$ . We consider the problem

$$(5.26) \quad \begin{cases} dY + AY dt = [F_1(Y) + F_2(t)]dt + G(t)dw_t, & \frac{T_{F_1, F_2, G, \xi}}{2} < t < \infty, \\ Y(\frac{T_{F_1, F_2, G, \xi}}{2}) = \bar{\xi}. \end{cases}$$

Thanks to Theorem 5.1, (5.26) has a local mild solution  $Y(t)$  for every  $T_{F_1, F_2, G, \xi} \leq t \leq \frac{3T_{F_1, F_2, G, \xi}}{2}$ . By the uniqueness of solution,  $X(t) = Y(t)$  a.s. for  $t \in [\frac{T_{F_1, F_2, G, \xi}}{2}, T_{F_1, F_2, G, \xi}]$ . This means that we have constructed a local mild solution to (5.1) on the interval  $[0, \frac{3T_{F_1, F_2, G, \xi}}{2}]$ . The independence of  $T_{F_1, F_2, G, \xi}$  with respect to  $C_{F_1, F_2, G, \xi}$  allows us to continue this procedure unlimitedly. Each time the local solution is extended over the fixed length  $\frac{T_{F_1, F_2, G, \xi}}{2}$  of interval. So, by finite times, the extended interval can cover the interval  $[0, T]$ .  $\square$



**5.2. Regularity for more regular initial data.** This subsection shows regularity of local mild solutions for more regular initial values. For any  $F_2 \in \mathcal{F}^{\gamma,\sigma}((0, T]; E)$ ,  $\max\{\beta, \frac{1}{2} - \eta\} < \gamma < \frac{1}{2}$ , and any initial value  $\xi \in \mathcal{D}(A^\gamma)$  satisfying  $\mathbb{E}|A^\gamma \xi|^2 < \infty$ , we can verify a stronger regularity than (5.3) for the local mild solution of (5.1).

**THEOREM 5.3.** *Let (4.2), (4.3), (H1) and (H3) be satisfied. Let  $F_2 \in \mathcal{F}^{\gamma,\sigma}((0, T]; E)$ ,  $\max\{\beta, \frac{1}{2} - \eta\} < \gamma < \frac{1}{2}$ , and  $\xi \in \mathcal{D}(A^\gamma)$  such that  $\mathbb{E}|A^\gamma \xi|^2 < \infty$ . Then (5.1) possesses a unique mild solution  $X$  in the function space:*

$$X \in \mathcal{C}((0, T_{F_1, F_2, G, \xi}]; \mathcal{D}(A^\eta)) \cap \mathcal{C}([0, T_{F_1, F_2, G, \xi}]; \mathcal{D}(A^\gamma)),$$

where  $T_{F_1, F_2, G, \xi}$  depends only on the squared norms  $|F_2|_{\mathcal{F}^{\beta,\sigma}}^2$ ,  $|G|_{\mathcal{F}^{\beta+\frac{1}{2},\sigma}}^2$  and  $\mathbb{E}|F_1(0)|^2$  and  $\mathbb{E}|A^\gamma \xi|^2$ . In addition,  $X$  satisfies the estimate

$$(5.27) \quad \mathbb{E}|X(t)|^2 + t^{2(\gamma-\beta)} \mathbb{E}|A^\gamma X(t)|^2 \leq C_{F_1, F_2, G, \xi}, \quad t \in [0, T_{F_1, F_2, G, \xi}]$$

with some constant  $C_{F_1, F_2, G, \xi}$  depending only on  $|F_2|_{\mathcal{F}^{\beta,\sigma}}^2$ ,  $|G|_{\mathcal{F}^{\beta+\frac{1}{2},\sigma}}^2$ ,  $\mathbb{E}|F_1(0)|^2$  and  $\mathbb{E}|A^\gamma \xi|^2$ .

**PROOF.** Since the embedding of  $\mathcal{D}(A^\gamma)$  in  $\mathcal{D}(A^\beta)$  is continuous, we have  $\xi \in \mathcal{D}(A^\beta)$  and  $\mathbb{E}|A^\beta \xi|^2 < \infty$ . In addition, due to (2.6), we also have  $F_2 \in \mathcal{F}^{\beta,\sigma}((0, T]; E)$ . Therefore, by Theorem 5.1, (5.1) possesses a unique mild solution  $X$  in the function space (5.3) which satisfies the estimate (5.4).

It now remains only to show that  $X \in \mathcal{C}([0, T_{F_1, F_2, G, \xi}]; \mathcal{D}(A^\gamma))$  and that  $X$  satisfies (5.27). For this purpose, we shall divide the proof into three steps. Throughout the proof,  $C_{F_1, F_2, G, \xi}$  denotes a universal constant which is determined in each occurrence by  $F_1, F_2, G$  and  $\xi$ .

**Step 1.** Let us verify that

$$(5.28) \quad \mathbb{E}|A^\eta X(t)|^2 \leq C_{F_1, F_2, G, \xi} t^{-2\varrho}, \quad t \in (0, T_{F_1, F_2, G, \xi}],$$

where  $\varrho = \max\{1 - \eta - \gamma, \eta - \beta\} \in (0, \frac{1}{2})$ . From (5.22), we have

$$\begin{aligned} A^\eta X(t) &= A^{\eta-\gamma} S(t) A^\gamma \xi + \int_0^t A^\eta S(t-s) [F_1(X(s)) + F_2(s)] ds \\ &\quad + \int_0^t A^\eta S(t-s) G(s) dw_s. \end{aligned}$$

Using (2.5) and (4.6), we then obtain that

$$\mathbb{E}|A^\eta X(t)|^2$$

$$\begin{aligned}
&\leq 4|A^{\eta-\gamma}S(t)|^2\mathbb{E}|A^\gamma\xi|^2 + 4\mathbb{E}\left|\int_0^t |A^\eta S(t-s)||F_1(X(s))|ds\right|^2 \\
&\quad + 4\mathbb{E}\left[\int_0^t |A^\eta S(t-s)||F_2(s)|ds\right]^2 + 4\mathbb{E}\left|\int_0^t A^\eta S(t-s)G(s)dw_s\right|^2 \\
&\leq C_\xi t^{\min\{-2(\eta-\gamma),0\}} + 4\iota_\eta^2\mathbb{E}\left[\int_0^t (t-s)^{-\eta}|F_1(X(s))|ds\right]^2 \\
&\quad + 4\iota_\eta^2|F_2|_{\mathcal{F}^{\gamma,\sigma}}^2\left[\int_0^t (t-s)^{\eta-1}s^{\gamma-1}ds\right]^2 \\
&\quad + 4c(E)\int_0^t |A^\eta S(t-s)|^2|G(s)|^2ds \\
&\leq C_\xi t^{\min\{-2(\eta-\gamma),0\}} + 4\iota_\eta^2t\int_0^t (t-s)^{-2\eta}\mathbb{E}|F_1(X(s))|^2ds \\
&\quad + 4\iota_\eta^2|F_2|_{\mathcal{F}^{\gamma,\sigma}}^2\mathbf{B}(\gamma,\eta)^2t^{2(\eta+\gamma-1)} + 4c(E)\iota_\eta^2|G|_{\mathcal{F}^{\beta+\frac{1}{2},\sigma}}^2\int_0^t (t-s)^{-2\eta}s^{2\beta-1}ds.
\end{aligned}$$

By (5.4) and (5.10), we have

$$\begin{aligned}
&4\iota_\eta^2t\int_0^t (t-s)^{-2\eta}\mathbb{E}|F_1(X(s))|^2ds \\
&\leq 12\iota_\eta^2t\int_0^t (t-s)^{-2\eta}[c_{F_1}^2\mathbb{E}|A^\eta X(s)|^2 + c_{F_1}^2\mathbb{E}|A^\beta X(s)|^2 + \mathbb{E}|F_1(0)|^2]ds \\
&\leq 12\iota_\eta^2c_{F_1}^2t\int_0^t (t-s)^{-2\eta}\mathbb{E}|A^\eta X(s)|^2ds \\
&\quad + \frac{12\iota_\eta^2[c_{F_1}^2C_{F_1,F_2,G,\xi} + \mathbb{E}|F_1(0)|^2]t^{2(1-\eta)}}{1-2\eta}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mathbb{E}|A^\eta X(t)|^2 \\
&\leq C_\xi t^{\min\{-2(\eta-\gamma),0\}} + 12\iota_\eta^2c_{F_1}^2t\int_0^t (t-s)^{-2\eta}\mathbb{E}|A^\eta X(s)|^2ds \\
&\quad + \frac{12\iota_\eta^2[c_{F_1}^2C_{F_1,F_2,G,\xi} + \mathbb{E}|F_1(0)|^2]t^{2(1-\eta)}}{1-2\eta} \\
&\quad + 4\iota_\eta^2|F_2|_{\mathcal{F}^{\gamma,\sigma}}^2\mathbf{B}(\gamma,\eta)^2t^{2(\eta+\gamma-1)} + 4c(E)\iota_\eta^2|G|_{\mathcal{F}^{\beta+\frac{1}{2},\sigma}}^2\int_0^t (t-s)^{-2\eta}s^{2\beta-1}ds \\
&\leq C_\xi t^{\min\{-2(\eta-\gamma),0\}} + C_{F_1,F_2,G,\xi}t^{-2\varrho} \\
&\quad + 12\iota_\eta^2c_{F_1}^2t\int_0^t (t-s)^{-2\eta}\mathbb{E}|A^\eta X(s)|^2ds,
\end{aligned}$$

(5.29)

$$\leq C_{F_1, F_2, G, \xi} t^{-2\varrho} + 12\iota_\eta^2 c_{F_1}^2 t \int_0^t (t-s)^{-2\eta} \mathbb{E}|A^\eta X(s)|^2 ds, \quad t \in (0, T_{F_1, F_2, G, \xi}],$$

here we used the estimates

$$\begin{cases} t^{2(\eta+\gamma-1)} \leq C t^{-2\varrho}, & t \in (0, T_{F_1, F_2, G, \xi}], \\ t^{2(\beta-\eta)} \leq C t^{-2\varrho}, & t \in (0, T_{F_1, F_2, G, \xi}], \\ t^{\min\{-2(\eta-\gamma), 0\}} \leq C t^{-2\varrho}, & t \in (0, T_{F_1, F_2, G, \xi}] \end{cases}$$

with some constant  $C$  depending only on  $T_{F_1, F_2, G, \xi}$ . Then the function  $q(t) = t^{2\varrho} \mathbb{E}|A^\eta X(t)|^2$  satisfies

$$(5.30) \quad q(t) \leq C_{F_1, F_2, G, \xi} + 12\iota_\eta^2 c_{F_1}^2 t^{1+2\varrho} \int_0^t (t-s)^{-2\eta} s^{-2\varrho} q(s) ds.$$

Let us solve the integral inequality as follows. Let  $\epsilon > 0$  denote a small parameter. For  $0 \leq t \leq \epsilon$  we have

$$\begin{aligned} q(t) &\leq C_{F_1, F_2, G, \xi} + 12\iota_\eta^2 c_{F_1}^2 t^{1+2\varrho} \int_0^t (t-s)^{-2\eta} s^{-2\varrho} ds \sup_{s \in [0, \epsilon]} q(s) \\ &= C_{F_1, F_2, G, \xi} + 12\iota_\eta^2 c_{F_1}^2 t^{2(1-\eta)} \mathbf{B}(1-2\varrho, 1-2\eta) \sup_{s \in [0, \epsilon]} q(s) \\ &\leq C_{F_1, F_2, G, \xi} + 12\iota_\eta^2 c_{F_1}^2 \epsilon^{2(1-\eta)} \mathbf{B}(1-2\varrho, 1-2\eta) \sup_{s \in [0, \epsilon]} q(s). \end{aligned}$$

Hence,

$$[1 - 12\iota_\eta^2 c_{F_1}^2 \epsilon^{2(1-\eta)} \mathbf{B}(1-2\varrho, 1-2\eta)] \sup_{s \in [0, \epsilon]} q(s) \leq C_{F_1, F_2, G, \xi}.$$

If  $\epsilon$  is taken sufficiently small so that  $12\iota_\eta^2 c_{F_1}^2 \epsilon^{2(1-\eta)} \mathbf{B}(1-2\varrho, 1-2\eta) \leq \frac{1}{2}$ , then we obtain that

$$(5.31) \quad \sup_{s \in [0, \epsilon]} q(s) \leq C_{F_1, F_2, G, \xi}.$$

Meanwhile, for  $\epsilon < t \leq T_{F_1, F_2, G, \xi}$  we have

$$\begin{aligned} q(t) &\leq C_{F_1, F_2, G, \xi} + 12\iota_\eta^2 c_{F_1}^2 t^{1+2\varrho} \int_0^\epsilon (t-s)^{-2\eta} s^{-2\varrho} ds \sup_{s \in [0, \epsilon]} q(s) \\ &\quad + 12\iota_\eta^2 c_{F_1}^2 t^{1+2\varrho} \int_\epsilon^t (t-s)^{-2\eta} \max\{\epsilon^{-2\varrho}, T^{-2\varrho}\} q(s) ds \end{aligned}$$

$$\leq C_{F_1, F_2, G, \xi} + 12\iota_\eta^2 c_{F_1}^2 \max\{\epsilon^{-2\varrho}, T^{-2\varrho}\} T^{1+2\varrho} \int_\epsilon^t (t-s)^{-2\eta} q(s) ds.$$

Lemma 2.12 then provides that

$$(5.32) \quad \sup_{t \in [\epsilon, T_{F_1, F_2, G, \xi}]} q(t) \leq C_{F_1, F_2, G, \xi}.$$

Thus, (5.28) follows from (5.31) and (5.32).

**Step 2.** Let us verify that  $X(t) \in \mathcal{D}(A^\gamma)$  for every  $t \in [0, T_{F_1, F_2, G, \xi}]$ . In virtue of (5.22), it suffices to show that the integrals  $\int_0^t A^\gamma S(t-s) F_2(s) ds$ ,  $\int_0^t A^\gamma S(t-s) G(s) dw_s$  and  $\int_0^t A^\gamma S(t-s) F_1(X(s)) ds$  are well-defined.

The first integral exists. Indeed, using (2.5) and (4.6), we have

$$(5.33) \quad \begin{aligned} \int_0^t |A^\gamma S(t-s) F_2(s)| ds &\leq \iota_\gamma |F_2|_{\mathcal{F}^{\gamma, \sigma}} \int_0^t (t-s)^{-\gamma} s^{\gamma-1} ds \\ &= \iota_\gamma |F_2|_{\mathcal{F}^{\gamma, \sigma}} \mathbf{B}(\gamma, 1-\gamma) < \infty, \quad t \in [0, T_{F_1, F_2, G, \xi}]. \end{aligned}$$

To show existence of the second integral, we have to verify that  $\int_0^t |A^\gamma S(t-s) G(s)|^2 ds < \infty$ . Indeed, by (2.5) and (4.6), we see that

$$(5.34) \quad \begin{aligned} \int_0^t |A^\gamma S(t-s) G(s)|^2 ds &\leq \int_0^t |A^\gamma S(t-s)|^2 |G(s)|^2 ds \\ &\leq \iota_\gamma^2 |G|_{\mathcal{F}^{\beta+\frac{1}{2}, \sigma}}^2 \int_0^t (t-s)^{-2\gamma} s^{2\beta-1} ds \\ &= \iota_\gamma^2 |G|_{\mathcal{F}^{\beta+\frac{1}{2}, \sigma}}^2 \mathbf{B}(2\beta, 1-2\gamma) t^{2(\beta-\gamma)} < \infty, \quad t \in (0, T_{F_1, F_2, G, \xi}]. \end{aligned}$$

We shall finish this step by showing that the last integral is well-defined. From (4.6), (5.4), (5.10) and (5.28), we observe that

$$(5.35) \quad \begin{aligned} &\mathbb{E} \int_0^t |A^\gamma S(t-s) F_1(X(s))|^2 ds \\ &\leq \int_0^t |A^\gamma S(t-s)|^2 \mathbb{E} |F_1(X(s))|^2 ds \\ &\leq 3\iota_\gamma^2 \int_0^t (t-s)^{-2\gamma} [c_{F_1}^2 \mathbb{E} |A^\eta X(s)|^2 + c_{F_1}^2 \mathbb{E} |A^\beta X(s)|^2 + \mathbb{E} |F_1(0)|^2] ds \\ &\leq 3\iota_\gamma^2 C_{F_1, F_2, G, \xi} \int_0^t (t-s)^{-2\gamma} [s^{-2\varrho} + 1] ds \\ &= 3\iota_\gamma^2 C_{F_1, F_2, G, \xi} \left[ \mathbf{B}(1-2\varrho, 1-2\gamma) t^{1-2\varrho-2\gamma} + \frac{t^{1-2\gamma}}{1-2\gamma} \right] \\ &< \infty, \quad t \in (0, T_{F_1, F_2, G, \xi}]. \end{aligned}$$

Consequently,  $\int_0^t |A^\gamma S(t-s)F_1(X(s))|^2 ds < \infty$  a.s. Therefore,

$$\int_0^t |A^\gamma S(t-s)F_1(X(s))| ds < \infty \quad \text{a.s.}$$

**Step 3.** Let us show the estimate (5.27). From (5.22), we observe that

$$\begin{aligned} & \mathbb{E}|A^\gamma X(t)|^2 \\ & \leq 4\mathbb{E}|A^\gamma S(t)\xi|^2 + 4\mathbb{E}\left[\int_0^t |A^\gamma S(t-s)F_1(X(s))| ds\right]^2 \\ & \quad + 4\left[\int_0^t |A^\gamma S(t-s)F_2(s)| ds\right]^2 + 4\mathbb{E}\left|\int_0^t A^\gamma S(t-s)G(s)dw_s\right|^2 \\ & \leq 4\mathbb{E}|A^\gamma S(t)\xi|^2 + 4t\mathbb{E}\int_0^t |A^\gamma S(t-s)F_1(X(s))|^2 ds \\ & \quad + 4\left[\int_0^t |A^\gamma S(t-s)F_2(s)| ds\right]^2 + 4c(E)\int_0^t |A^\gamma S(t-s)G(s)|^2 ds. \end{aligned}$$

Using (4.8), (5.33), (5.34) and (5.35), we have

$$\begin{aligned} & \mathbb{E}|A^\gamma X(t)|^2 \\ & \leq 4\iota_0^2 e^{-2\nu t} \mathbb{E}|A^\gamma \xi|^2 + C_{F_1, F_2, G, \xi} \left[ \mathbf{B}(1-2\varrho, 1-2\gamma) t^{2(1-\varrho-\gamma)} + \frac{t^{2(1-\gamma)}}{1-2\gamma} \right] \\ & \quad + 4\iota_\gamma^2 |F_2|_{\mathcal{F}^{\gamma, \sigma}}^2 \mathbf{B}(\gamma, 1-\gamma)^2 + 4c(E) \iota_\gamma^2 |G|_{\mathcal{F}^{\beta+\frac{1}{2}, \sigma}}^2 \mathbf{B}(2\beta, 1-2\gamma) t^{2(\beta-\gamma)} \\ & \leq C_{F_1, F_2, G, \xi} t^{-2(\gamma-\beta)}, \quad t \in (0, T_{F_1, F_2, G, \xi}], \end{aligned}$$

here we used the estimates

$$\begin{cases} 4\iota_0^2 \mathbb{E}|A^\gamma \xi|^2 e^{-2\nu t} \leq C t^{-2(\gamma-\beta)}, & t \in (0, T_{F_1, F_2, G, \xi}], \\ t^{2(1-\varrho-\gamma)} \leq C t^{-2(\gamma-\beta)}, & t \in (0, T_{F_1, F_2, G, \xi}], \\ t^{2(1-\gamma)} < C t^{-2(\gamma-\beta)}, & t \in (0, T_{F_1, F_2, G, \xi}], \\ 4\iota_\gamma^2 |F_2|_{\mathcal{F}^{\gamma, \sigma}}^2 \mathbf{B}(\gamma, 1-\gamma)^2 < C t^{-2(\gamma-\beta)}, & t \in (0, T_{F_1, F_2, G, \xi}] \end{cases}$$

with some constant  $C$  depending only on  $T_{F_1, F_2, G, \xi}$ . Thus,

$$t^{2(\gamma-\beta)} \mathbb{E}|A^\gamma X(t)|^2 \leq C_{F_1, F_2, G, \xi}, \quad t \in [0, T_{F_1, F_2, G, \xi}].$$

This, together with (5.4), then derives (5.27).

**Step 4.** Let us verify that  $A^\gamma X \in \mathcal{C}([0, T_{F_1, F_2, G, \xi}]; E)$ . Similarly to (5.10), for  $t \in (0, T_{F_1, F_2, G, \xi}]$  we have

$$\mathbb{E}|F_1(X(t))|^2 \leq 3[c_{F_1}^2 \mathbb{E}|A^\eta X(t)|^2 + c_{F_1}^2 \mathbb{E}|A^\beta X(t)|^2 + \mathbb{E}|F_1(0)|^2].$$

Due to (5.4) and (5.28), we observe that

$$(5.36) \quad \begin{aligned} \mathbb{E}|F_1(X(t))|^2 &\leq C_{F_1, F_2, G, \xi}(t^{-2\varrho} + 1) \\ &\leq C_{F_1, F_2, G, \xi}t^{-2\varrho}, \quad t \in (0, T_{F_1, F_2, G, \xi}]. \end{aligned}$$

First, we shall show that  $A^\gamma X \in \mathcal{C}((0, T_{F_1, F_2, G, \xi}]; E)$ . By repeating the same argument as in verifying (5.16) in **Step 1** of the proof of Theorem 5.1, for every  $\rho \in (\frac{1}{2}, 1 - \gamma)$  we see that

$$\begin{aligned} &\mathbb{E}|A^\gamma[X(t) - X(s)]|^2 \\ &\leq \frac{4t_{1-\rho}^2 \iota_{\gamma+\rho-\beta}^2}{\rho^2} \mathbb{E}|A^\beta \xi|^2 s^{2(\beta-\gamma-\rho)} (t-s)^{2\rho} \\ &\quad + 4 \left[ \frac{\iota_{1-\rho} \iota_{\gamma+\rho} \mathbf{B}(\beta, 1-\gamma-\rho)}{\rho} + \iota_\gamma \mathbf{B}(\gamma+\rho, 1-\gamma) \right]^2 \\ &\quad \times |F_2|_{\mathcal{F}^{\beta, \sigma}}^2 s^{2(\beta-\gamma-\rho)} (t-s)^{2\rho} \\ &\quad + \frac{4t_{1-\rho}^2 \iota_{\gamma+\rho}^2 (1-\gamma-\rho)}{\rho^2} (t-s)^{2\rho} s^{1-\gamma-\rho} \int_0^s (s-r)^{-\gamma-\rho} \mathbb{E}|F_1(X(r))|^2 dr \\ &\quad + 4t_\gamma^2 (t-s) \int_s^t (t-r)^{-2\gamma} \mathbb{E}|F_1(X(r))|^2 dr. \end{aligned}$$

Using (5.36), we obtain that

$$\begin{aligned} &\mathbb{E}|A^\gamma[X(t) - X(s)]|^2 \\ &\leq \frac{4t_{1-\rho}^2 \iota_{\gamma+\rho-\beta}^2}{\rho^2} \mathbb{E}|A^\beta \xi|^2 s^{2(\beta-\gamma-\rho)} (t-s)^{2\rho} \\ &\quad + 4 \left[ \frac{\iota_{1-\rho} \iota_{\gamma+\rho} \mathbf{B}(\beta, 1-\gamma-\rho)}{\rho} + \iota_\gamma \mathbf{B}(\gamma+\rho, 1-\gamma) \right]^2 \\ &\quad \times |F_2|_{\mathcal{F}^{\beta, \sigma}}^2 s^{2(\beta-\gamma-\rho)} (t-s)^{2\rho} \\ &\quad + C_{F_1, F_2, G, \xi} (t-s)^{2\rho} s^{1-\gamma-\rho} \int_0^s (s-r)^{-\gamma-\rho} r^{-2\varrho} dr \\ &\quad + C_{F_1, F_2, G, \xi} (t-s) \int_s^t (t-r)^{-2\gamma} r^{-2\varrho} dr \\ &\leq C_{F_1, F_2, G, \xi} s^{2(\beta-\gamma-\rho)} (t-s)^{2\rho} \\ &\quad + C_{F_1, F_2, G, \xi} s^{2(1-\gamma-\rho-\varrho)} (t-s)^{2\rho} \\ &\quad + C_{F_1, F_2, G, \xi} (t-s) \int_s^t (t-r)^{-2\gamma} r^{-2\varrho} dr. \end{aligned}$$

Let us estimate the latter integral. Fix  $\epsilon \in (0, \min\{1 - 2\gamma, 2\varrho\})$ . Since

$$r^{-2\varrho} = r^{-\epsilon} r^{\epsilon-2\varrho} < (r-s)^{-\epsilon} s^{\epsilon-2\varrho}, \quad r \in (s, t),$$

we have

$$\begin{aligned} \int_s^t (t-r)^{-2\gamma} r^{-2\varrho} dr &\leq s^{\epsilon-2\varrho} \int_s^t (t-r)^{-2\gamma} (r-s)^{-\epsilon} dr \\ &= \mathbf{B}(1-\epsilon, 1-2\gamma) s^{\epsilon-2\varrho} (t-s)^{1-2\gamma-\epsilon}. \end{aligned}$$

Hence,

$$\begin{aligned} &\mathbb{E}|A^\gamma[X(t) - X(s)]|^2 \\ &\leq C_{F_1, F_2, G, \xi} [s^{2(\beta-\gamma-\rho)}(t-s)^{2\rho} + s^{2(1-\gamma-\rho-\varrho)}(t-s)^{2\rho} + s^{\epsilon-2\varrho}(t-s)^{2-2\gamma-\epsilon}]. \end{aligned}$$

Since  $2\rho > 1$  and  $2-2\gamma-\epsilon > 1$ , the Kolmogorov test then provides that  $A^\gamma X \in \mathcal{C}((0, T_{F_1, F_2, G, \xi}]; E)$ .

Now, we shall verify that  $A^\gamma X$  is continuous at  $t = 0$ . By using (5.36) (instead of (5.11)), we will repeat the same argument as in showing the continuity of  $A^\beta \Psi Y$  at  $t = 0$  in **Step 1** of the proof of Theorem 5.1. We then obtain the continuity of  $A^\gamma X$  at  $t = 0$ . Thus, it is concluded that  $A^\gamma X \in \mathcal{C}([0, T_{F_1, F_2, G, \xi}]; E)$ .  $\square$

**5.3. Regular dependence of solutions on initial data.** Let  $\mathcal{B}_1$  and  $\mathcal{B}_2$  be bounded balls

$$(5.37) \quad \mathcal{B}_1 = \{f \in \mathcal{F}^{\beta, \sigma}((0, T]; E) : |f|_{\mathcal{F}^{\beta, \sigma}} \leq R_1\}, \quad 0 < R_1 < \infty,$$

$$(5.38) \quad \mathcal{B}_2 = \{f \in \mathcal{F}^{\beta+\frac{1}{2}, \sigma}((0, T]; E) : |f|_{\mathcal{F}^{\beta+\frac{1}{2}, \sigma}} \leq R_2\}, \quad 0 < R_2 < \infty$$

of the spaces  $\mathcal{F}^{\beta, \sigma}((0, T]; E)$  and  $\mathcal{F}^{\beta+\frac{1}{2}, \sigma}((0, T]; E)$ , respectively. And let  $B_A$  be a set of random variable

$$(5.39) \quad B_A = \{\xi : \xi \in \mathcal{D}(A^\beta) \text{ a.s. and } \mathbb{E}|A^\beta \xi|^2 \leq R_3^2\}, \quad 0 < R_3 < \infty.$$

According to Theorem 5.1, for every  $F_2 \in \mathcal{B}_1, G \in \mathcal{B}_2$  and  $\xi \in B_A$ , there exists a local solution of (5.1) on some interval  $[0, T_{F_1, F_2, G, \xi}]$ . Furthermore, by virtue of **Step 1** and **Step 2** of the proof of Theorem 5.1,

$$(5.40) \quad \begin{aligned} &\text{there is a time } T_{\mathcal{B}_1, \mathcal{B}_2, B_A} > 0 \text{ such that} \\ &[0, T_{\mathcal{B}_1, \mathcal{B}_2, B_A}] \subset [0, T_{F_1, F_2, G, \xi}] \text{ for all } (F_2, G, \xi) \in \mathcal{B}_1 \times \mathcal{B}_2 \times B_A. \end{aligned}$$

Indeed, in view of (5.14), (5.15) and (5.21),  $T_{F_1, F_2, G, \xi}$  can be chosen to be any time  $S$  satisfying the conditions

$$18\iota_\beta^2 c_{F_1}^2 \kappa^2 \mathbf{B}(1+2\beta-2\eta, 1-2\beta) S^{2(1+\beta-2\eta)}$$

$$\begin{aligned}
& + \frac{18\iota_\beta^2 [c_{F_1}^2 \kappa^2 + \mathbb{E}|F_1(0)|^2]}{1-2\beta} S^{2(1-\beta)} \leq \frac{\kappa^2}{2}, \\
& 18\iota_\beta^2 c_{F_1}^2 \kappa^2 \mathbf{B}(1+2\beta-2\eta, 1-2\beta) S^{2(1+\beta-2\eta)} \\
& + \frac{18\iota_\beta^2 [c_{F_1}^2 \kappa^2 + \mathbb{E}|F_1(0)|^2]}{1-2\beta} S^{2(1-\beta)} \leq \frac{\kappa^2}{2},
\end{aligned}$$

and

$$\begin{aligned}
& 2c_{F_1}^2 \left[ \iota_\eta^2 \mathbf{B}(1+2\beta-2\eta, 1-2\eta) + \iota_\beta^2 \mathbf{B}(1+2\beta-2\eta, 1-2\beta) \right. \\
& \left. + \frac{\iota_\eta^2 S^{2(\eta-\beta)}}{1-2\eta} + \frac{\iota_\beta^2 S^{2(\eta-\beta)}}{1-2\beta} \right] S^{2(1-\eta)} < 1,
\end{aligned}$$

where  $\kappa$  is defined by (5.7) and (5.13). Consequently, we can choose  $T_{F_1, F_2, G, \xi}$  such that it depends continuously on the norms  $|F_2|_{\mathcal{F}^{\beta, \sigma}}, |G|_{\mathcal{F}^{\beta+\frac{1}{2}, \sigma}}$  and  $\mathbb{E}|A^\beta \xi|^2$ . This implies (5.40).

We shall show continuous dependence of solutions on  $(F_2, G, \xi)$  in the sense specified in the following theorem.

**THEOREM 5.4.** *Let (4.2), (4.3), (H1), (H2) and (H3) be satisfied. Let  $X$  and  $\bar{X}$  be the solutions of (5.1) for the data  $(F_2, G, \xi)$  and  $(\bar{F}_2, \bar{G}, \bar{\xi})$  in  $\mathcal{B}_1 \times \mathcal{B}_2 \times B_A$ , respectively. Then there exists a constant  $C_{\mathcal{B}_1, \mathcal{B}_2, B_A}$  depending only on  $\mathcal{B}_1, \mathcal{B}_2$  and  $B_A$  such that*

$$\begin{aligned}
(5.41) \quad & t^{2\eta} \mathbb{E}|A^\eta[X(s) - \bar{X}(s)]|^2 + t^{2\eta} \mathbb{E}|A^\beta[X(s) - \bar{X}(s)]|^2 + \mathbb{E}|X(t) - \bar{X}(t)|^2 \\
& \leq C_{\mathcal{B}_1, \mathcal{B}_2, B_A} [\mathbb{E}|\xi - \bar{\xi}|^2 + t^{2\beta} |F_2 - \bar{F}_2|_{\mathcal{F}^{\beta, \sigma}}^2 + t^{2\beta} |G - \bar{G}|_{\mathcal{F}^{\beta+\frac{1}{2}, \sigma}}^2],
\end{aligned}$$

and

$$\begin{aligned}
(5.42) \quad & t^{2(\eta-\beta)} [\mathbb{E}|A^\eta[X(s) - \bar{X}(s)]|^2 + \mathbb{E}|A^\beta[X(s) - \bar{X}(s)]|^2] \\
& \leq C_{\mathcal{B}_1, \mathcal{B}_2, B_A} [\mathbb{E}|A^\beta(\xi - \bar{\xi})|^2 + |F_2 - \bar{F}_2|_{\mathcal{F}^{\beta, \sigma}}^2 + |G - \bar{G}|_{\mathcal{F}^{\beta+\frac{1}{2}, \sigma}}^2]
\end{aligned}$$

for every  $t \in (0, T_{\mathcal{B}_1, \mathcal{B}_2, B_A}]$ .

**PROOF.** This theorem is proved by analogous arguments as in the proof of Theorem 5.1.

Indeed, let us first verify (5.41). If  $\mathbb{E}|\xi - \bar{\xi}|^2 = \infty$ , then the conclusion is obvious. Therefore, we may assume that  $\mathbb{E}|\xi - \bar{\xi}|^2 < \infty$ .

First, we shall give an estimate for

$$t^{2\eta} \mathbb{E}[|A^\eta[X(s) - \bar{X}(s)]|^2 + |A^\beta[X(s) - \bar{X}(s)]|^2].$$



For  $\theta \in [0, \frac{1}{2})$  and  $0 < t \leq T_{\mathcal{B}_1, \mathcal{B}_2, B_A}$ , by using (2.5), (4.6) and (5.2), we observe that

$$\begin{aligned}
& t^\theta |A^\theta[X(t) - \bar{X}(t)]| \\
&= \left| t^\theta A^\theta S(t)(\xi - \bar{\xi}) + \int_0^t t^\theta A^\theta S(t-s)[F_1(X(s)) - F_1(\bar{X}(s))]ds \right. \\
&\quad \left. + \int_0^t t^\theta A^\theta S(t-s)[F_2(s) - \bar{F}_2(s)]ds + \int_0^t t^\theta A^\theta S(t-s)[G(s) - \bar{G}(s)]dw_s \right| \\
&\leq \iota_\theta |\xi - \bar{\xi}| \\
&\quad + \iota_\theta c_{F_1} \int_0^t t^\theta (t-s)^{-\theta} [|A^\eta[X(s) - \bar{X}(s)]| + |A^\beta[X(s) - \bar{X}(s)]|]ds \\
&\quad + \iota_\theta |F_2 - \bar{F}_2|_{\mathcal{F}^{\beta, \sigma}} \int_0^t t^\theta (t-s)^{-\theta} s^{\beta-1} ds \\
&\quad + \left| \int_0^t t^\theta A^\theta S(t-s)[G(s) - \bar{G}(s)]dw_s \right| \\
&= \iota_\theta |\xi - \bar{\xi}| + \iota_\theta |F_2 - \bar{F}_2|_{\mathcal{F}^{\beta, \sigma}} \mathbf{B}(\beta, 1-\theta) t^\beta \\
&\quad + \iota_\theta c_{F_1} \int_0^t t^\theta (t-s)^{-\theta} [|A^\eta[X(s) - \bar{X}(s)]| + |A^\beta[X(s) - \bar{X}(s)]|]ds \\
&\quad + \left| \int_0^t t^\theta A^\theta S(t-s)[G(s) - \bar{G}(s)]dw_s \right|.
\end{aligned}$$

Thus,

$$\begin{aligned}
& \mathbb{E} |t^\theta A^\theta[X(t) - \bar{X}(t)]|^2 \\
&\leq 4\iota_\theta^2 \mathbb{E} |\xi - \bar{\xi}|^2 + 4\iota_\theta^2 |F_2 - \bar{F}_2|_{\mathcal{F}^{\beta, \sigma}}^2 \mathbf{B}(\beta, 1-\theta)^2 t^{2\beta} \\
&\quad + 4\iota_\theta^2 c_{F_1}^2 t^{2\theta} \mathbb{E} \left[ \int_0^t (t-s)^{-\theta} [|A^\eta[X(s) - \bar{X}(s)]| + |A^\beta[X(s) - \bar{X}(s)]|]ds \right]^2 \\
&\quad + 4\mathbb{E} \left| \int_0^t t^\theta A^\theta S(t-s)[G(s) - \bar{G}(s)]dw_s \right|^2 \\
&\leq 4\iota_\theta^2 \mathbb{E} |\xi - \bar{\xi}|^2 + 4\iota_\theta^2 |F_2 - \bar{F}_2|_{\mathcal{F}^{\beta, \sigma}}^2 \mathbf{B}(\beta, 1-\theta)^2 t^{2\beta} \\
&\quad + 4\iota_\theta^2 c_{F_1}^2 t^{2\theta+1} \int_0^t (t-s)^{-2\theta} \mathbb{E} [|A^\eta[X(s) - \bar{X}(s)]| + |A^\beta[X(s) - \bar{X}(s)]|]^2 ds \\
&\quad + 4c(E) \iota_\theta^2 |G - \bar{G}|_{\mathcal{F}^{\beta+\frac{1}{2}, \sigma}}^2 \int_0^t t^{2\theta} (t-s)^{-2\theta} s^{2\beta-1} ds \\
&\quad (5.43) \\
&\leq 4\iota_\theta^2 \mathbb{E} |\xi - \bar{\xi}|^2 + 4\iota_\theta^2 \mathbf{B}(\beta, 1-\theta)^2 t^{2\beta} |F_2 - \bar{F}_2|_{\mathcal{F}^{\beta, \sigma}}^2 \\
&\quad + 4c(E) \iota_\theta^2 \mathbf{B}(2\beta, 1-2\theta) t^{2\beta} |G - \bar{G}|_{\mathcal{F}^{\beta+\frac{1}{2}, \sigma}}^2
\end{aligned}$$

$$+ 8\iota_\theta^2 c_{F_1}^2 t^{2\theta+1} \int_0^t (t-s)^{-2\theta} \mathbb{E}[|A^\eta[X(s) - \bar{X}(s)]|^2 + |A^\beta[X(s) - \bar{X}(s)]|^2] ds.$$

Applying these estimates with  $\theta = \beta$  and  $\theta = \eta$ , we have

$$\begin{aligned} & \mathbb{E}|A^\beta[X(t) - \bar{X}(t)]|^2 \\ & \leq 4\iota_\beta^2 \mathbb{E}|\xi - \bar{\xi}|^2 t^{-2\beta} + 4\iota_\beta^2 \mathbf{B}(\beta, 1 - \beta)^2 |F_2 - \bar{F}_2|_{\mathcal{F}^{\beta, \sigma}}^2 \\ & \quad + 4c(E)\iota_\beta^2 \mathbf{B}(2\beta, 1 - 2\beta)^2 |G - \bar{G}|_{\mathcal{F}^{\beta+\frac{1}{2}, \sigma}}^2 \\ & \quad + 8\iota_\beta^2 c_{F_1}^2 t \int_0^t (t-s)^{-2\beta} \mathbb{E}[|A^\eta[X(s) - \bar{X}(s)]|^2 + |A^\beta[X(s) - \bar{X}(s)]|^2] ds, \end{aligned}$$

and

$$\begin{aligned} & t^{2\eta} \mathbb{E}|A^\eta[X(t) - \bar{X}(t)]|^2 \\ & \leq 4\iota_\eta^2 \mathbb{E}|\xi - \bar{\xi}|^2 + 4\iota_\eta^2 \mathbf{B}(\beta, 1 - \eta)^2 t^{2\beta} |F_2 - \bar{F}_2|_{\mathcal{F}^{\beta, \sigma}}^2 \\ & \quad + 4c(E)\iota_\eta^2 \mathbf{B}(2\beta, 1 - 2\eta)^2 t^{2\beta} |G - \bar{G}|_{\mathcal{F}^{\beta+\frac{1}{2}, \sigma}}^2 \\ & \quad + 8\iota_\eta^2 c_{F_1}^2 t^{2\eta+1} \int_0^t (t-s)^{-2\eta} \mathbb{E}[|A^\eta[X(s) - \bar{X}(s)]|^2 + |A^\beta[X(s) - \bar{X}(s)]|^2] ds. \end{aligned}$$

By putting

$$q(t) = t^{2\eta} \mathbb{E}[|A^\eta[X(s) - \bar{X}(s)]|^2 + |A^\beta[X(s) - \bar{X}(s)]|^2],$$

we then obtain an integral inequality

$$\begin{aligned} q(t) & \leq 4(\iota_\beta^2 t^{2(\eta-\beta)} + \iota_\eta^2) \mathbb{E}|\xi - \bar{\xi}|^2 \\ & \quad + 4[\iota_\beta^2 \mathbf{B}(\beta, 1 - \beta)^2 t^{2\eta} + \iota_\eta^2 \mathbf{B}(\beta, 1 - \eta)^2 t^{2\beta}] |F_2 - \bar{F}_2|_{\mathcal{F}^{\beta, \sigma}}^2 \\ & \quad + 4c(E)[\iota_\beta^2 \mathbf{B}(2\beta, 1 - 2\beta)^2 t^{2\eta} + \iota_\eta^2 \mathbf{B}(2\beta, 1 - 2\eta)^2 t^{2\beta}] |G - \bar{G}|_{\mathcal{F}^{\beta+\frac{1}{2}, \sigma}}^2 \\ & \quad + 8c_{F_1}^2 t^{2\eta+1} \int_0^t [\iota_\beta^2 (t-s)^{-2\beta} + \iota_\eta^2 (t-s)^{-2\eta}] s^{-2\eta} q(s) ds \\ (5.44) \quad & \leq C_{\mathcal{B}_1, \mathcal{B}_2, B_A} [\mathbb{E}|\xi - \bar{\xi}|^2 + t^{2\beta} |F_2 - \bar{F}_2|_{\mathcal{F}^{\beta, \sigma}}^2 + t^{2\beta} |G - \bar{G}|_{\mathcal{F}^{\beta+\frac{1}{2}, \sigma}}^2] \\ & \quad + 8c_{F_1}^2 t^{2\eta+1} \int_0^t [\iota_\beta^2 (t-s)^{-2\beta} + \iota_\eta^2 (t-s)^{-2\eta}] s^{-2\eta} q(s) ds \end{aligned}$$

for  $0 < t \leq T_{\mathcal{B}_1, \mathcal{B}_2, B_A}$ . We use the same techniques as in solving the integral inequality (5.30) to solve (5.44). Arguing first in a small interval  $[0, \epsilon]$  and then in the other interval  $[\epsilon, T_{\mathcal{B}_1, \mathcal{B}_2, B_A}]$ , we obtain that

$$t^{2\eta} \mathbb{E}[|A^\eta[X(s) - \bar{X}(s)]|^2 + |A^\beta[X(s) - \bar{X}(s)]|^2] = q(t)$$

$$(5.45) \quad \leq C_{\mathcal{B}_1, \mathcal{B}_2, B_A} [\mathbb{E}|\xi - \bar{\xi}|^2 + t^{2\beta}|F_2 - \bar{F}_2|_{\mathcal{F}^{\beta, \sigma}}^2 + t^{2\beta}|G - \bar{G}|_{\mathcal{F}^{\beta+\frac{1}{2}, \sigma}}^2]$$

for every  $t \in (0, T_{\mathcal{B}_1, \mathcal{B}_2, B_A}]$ .

Now, we shall give an estimate for  $\mathbb{E}|X(t) - \bar{X}(t)|^2$ . Taking  $\theta = 0$  in (5.43), we have

$$\begin{aligned} \mathbb{E}|X(t) - \bar{X}(t)|^2 &\leq 4\iota_0 \mathbb{E}|\xi - \bar{\xi}|^2 + 4\iota_0 \mathbf{B}(\beta, 1)^2 t^{2\beta}|F_2 - \bar{F}_2|_{\mathcal{F}^{\beta, \sigma}}^2 \\ &\quad + 4c(E) \mathbf{B}(2\beta, 1) t^{2\beta}|G - \bar{G}|_{\mathcal{F}^{\beta+\frac{1}{2}, \sigma}}^2 + 8\iota_0^2 c_{F_1}^2 t \int_0^t s^{-2\eta} q(s) ds. \end{aligned}$$

Using (5.45), we observe that

$$\begin{aligned} &t \int_0^t s^{-2\eta} q(s) ds \\ &\leq C_{\mathcal{B}_1, \mathcal{B}_2, B_A} t \int_0^t s^{-2\eta} [\mathbb{E}|\xi - \bar{\xi}|^2 + s^{2\beta}|F_2 - \bar{F}_2|_{\mathcal{F}^{\beta, \sigma}}^2 + s^{2\beta}|G - \bar{G}|_{\mathcal{F}^{\beta+\frac{1}{2}, \sigma}}^2] ds \\ &\leq \frac{C_{\mathcal{B}_1, \mathcal{B}_2, B_A} t^{2(1-\eta)} \mathbb{E}|\xi - \bar{\xi}|^2}{1 - 2\eta} \\ &\quad + \frac{C_{\mathcal{B}_1, \mathcal{B}_2, B_A} t^{2(1+\beta-\eta)} [|F_2 - \bar{F}_2|_{\mathcal{F}^{\beta, \sigma}}^2 + |G - \bar{G}|_{\mathcal{F}^{\beta+\frac{1}{2}, \sigma}}^2]}{1 + 2\beta - 2\eta}. \end{aligned}$$

Therefore,

$$(5.46) \quad \begin{aligned} \mathbb{E}|X(t) - \bar{X}(t)|^2 \\ \leq C_{\mathcal{B}_1, \mathcal{B}_2, B_A} [\mathbb{E}|\xi - \bar{\xi}|^2 + t^{2\beta}|F_2 - \bar{F}_2|_{\mathcal{F}^{\beta, \sigma}}^2 + t^{2\beta}|G - \bar{G}|_{\mathcal{F}^{\beta+\frac{1}{2}, \sigma}}^2] \end{aligned}$$

for  $t \in (0, T_{\mathcal{B}_1, \mathcal{B}_2, B_A}]$ . By (5.45) and (5.46), (5.41) has been verified.

Let us now show (5.42). By substituting the estimate

$$|A^\theta S(t)(\xi - \bar{\xi})| \leq \iota_{\beta-\theta} t^{\beta-\theta} |A^\beta(\xi - \bar{\xi})|$$

with  $\theta = \beta$  and  $\theta = \eta$  for  $|A^\theta S(t)(\xi - \bar{\xi})| \leq \iota_\theta t^{-\theta} |\xi - \bar{\xi}|$ , we obtain a similar result to (5.43):

$$\begin{aligned} &\mathbb{E}|t^\theta A^\theta [X(t) - \bar{X}(t)]|^2 \\ &\leq 4\iota_{\beta-\theta}^2 t^{2(\beta-\theta)} \mathbb{E}|A^\beta(\xi - \bar{\xi})|^2 + 4\iota_\theta^2 \mathbf{B}(\beta, 1 - \theta)^2 t^{2\beta}|F_2 - \bar{F}_2|_{\mathcal{F}^{\beta, \sigma}}^2 \\ &\quad + 4c(E) \iota_\theta^2 \mathbf{B}(2\beta, 1 - 2\theta) t^{2\beta}|G - \bar{G}|_{\mathcal{F}^{\beta+\frac{1}{2}, \sigma}}^2 \\ &\quad + 8\iota_\theta^2 c_{F_1}^2 t^{2\theta+1} \int_0^t (t-s)^{-2\theta} \mathbb{E}[|A^\eta[X(s) - \bar{X}(s)]|^2 + |A^\beta[X(s) - \bar{X}(s)]|^2] ds. \end{aligned}$$

Using the same arguments as in verifying (5.45), we conclude that (5.42) holds true. It completes the proof.  $\square$

5.4. *Case  $\tilde{\beta} = 0$ .* This subsection investigates the critical case of the Lipschitz condition (5.2) when  $\tilde{\beta} = 0$ . We assume that

(H1')  $F_1$  defines on the domain  $\mathcal{D}(A^\eta)$  and (5.2) is valid with  $\tilde{\beta} = 0$ , i.e.

$$(5.47) \quad |F_1(x) - F_1(y)| \leq c_{F_1}[|A^\eta(x - y)| + |x - y|], \quad x, y \in \mathcal{D}(A^\eta).$$

We can then generalize some results of Theorem 5.1.

**THEOREM 5.5.** *Let (4.2), (4.3), (H1'), (H2) and (H3) be satisfied. Assume that  $\mathbb{E}|\xi|^2 < \infty$ . Then (5.1) possesses a unique mild solution  $X$  in the function space:*

$$X \in \mathcal{C}((0, T_{F_1, F_2, G, \xi}]; \mathcal{D}(A^\eta)),$$

where  $T_{F_1, F_2, G, \xi}$  depends only on the squared norms  $|F_2|_{\mathcal{F}^{\beta, \sigma}}^2, |G|_{\mathcal{F}^{\beta + \frac{1}{2}, \sigma}}^2$  and  $\mathbb{E}|F_1(0)|^2$ . In addition,  $X$  satisfies the estimate

$$(5.48) \quad \mathbb{E}|X(t)|^2 \leq C_{F_1, F_2, G, \xi}, \quad t \in [0, T_{F_1, F_2, G, \xi}]$$

with some constant  $C_{F_1, F_2, G, \xi}$  depending only on  $|F_2|_{\mathcal{F}^{\beta, \sigma}}^2, |G|_{\mathcal{F}^{\beta + \frac{1}{2}, \sigma}}^2, \mathbb{E}|F_1(0)|^2$  and  $\mathbb{E}|\xi|^2$ .

**PROOF.** The proof is analogous to that of Theorem 5.1. For each  $S \in (0, T]$ , we set the Banach space:

$$\Xi(S) = \{Y \in \mathcal{C}((0, S]; \mathcal{D}(A^\eta)) \cap \mathcal{C}([0, S]; E) \text{ such that} \\ \sup_{0 < t \leq S} t^{2\eta} \mathbb{E}|A^\eta Y(t)|^2 + \sup_{0 \leq t \leq S} \mathbb{E}|Y(t)|^2 < \infty\}$$

with norm

$$|Y|_{\Xi(S)} = \left[ \sup_{0 < t \leq S} t^{2\eta} \mathbb{E}|A^\eta Y(t)|^2 + \sup_{0 \leq t \leq S} \mathbb{E}|Y(t)|^2 \right]^{\frac{1}{2}}.$$

Consider a nonempty closed subset  $\Upsilon(S)$  of  $\Xi(S)$  which consists of all function  $Y \in \Xi(S)$  such that

$$(5.49) \quad \max\left\{ \sup_{0 < t \leq S} t^{2\eta} \mathbb{E}|Y(t)|^2, \sup_{0 \leq t \leq S} \mathbb{E}|A^\beta Y(t)|^2 \right\} \leq \kappa^2$$

with  $\kappa > 0$  which will be fixed appropriately.

Similarly to the proof of Theorem 5.1, if we choose  $\kappa > 0$  dependent only on  $|F_2|_{\mathcal{F}^{\beta, \sigma}}^2, |G|_{\mathcal{F}^{\beta + \frac{1}{2}, \sigma}}^2, \mathbb{E}|F_1(0)|^2$  and  $\mathbb{E}|A^\beta \xi|^2$ , and if  $S$  is sufficiently small, then the mapping  $\Phi$  defined by (5.9) maps the set  $\Upsilon(S)$  into itself and is

contraction with respect to the norm of  $\Xi(S)$ . Consequently,  $\Phi$  possesses a unique fixed point  $X \in \Upsilon(S)$ , i.e. for every  $t \in [0, S]$ ,  $X(t) = \Phi X(t)$  a.s. This means that  $X$  is a local mild solution of (5.1). Following **Step 4** and **Step 6** in the proof of Theorem 5.1, the estimate (5.48) and uniqueness of the solution are verified.  $\square$

By the same arguments as in Corollary 5.2, global mild solutions of (5.1) can be constructed.

**COROLLARY 5.6** (global existence). *Assume that in Theorem 5.5 the constant  $C_{F_1, F_2, G, \xi}$  is independent of  $T_{F_1, F_2, G, \xi}$  for every initial value  $\xi \in E$ . Then (5.1) possesses a unique mild solution on the interval  $[0, T]$ .*

The next theorem shows regularity of local mild solutions for more regular initial values. The proof of the theorem is similar to that of Theorem 5.3, so we omit it.

**THEOREM 5.7.** *Let (4.2), (4.3), (H1'), (H2) and (H3) be satisfied. Let  $F_2 \in \mathcal{F}^{\gamma, \sigma}((0, T]; E)$ ,  $\max\{\beta, \frac{1}{2} - \eta\} < \gamma < \frac{1}{2}$ , and  $\xi \in \mathcal{D}(A^\gamma)$  such that  $\mathbb{E}|A^\gamma \xi|^2 < \infty$ . Then (5.1) possesses a unique mild solution  $X$  in the function space:*

$$X \in \mathcal{C}((0, T_{F_1, F_2, G, \xi}]; \mathcal{D}(A^\eta)) \cap \mathcal{C}([0, T_{F_1, F_2, G, \xi}]; \mathcal{D}(A^\gamma)),$$

where  $T_{F_1, F_2, G, \xi}$  depends only on the squared norms  $|F_2|_{\mathcal{F}^{\beta, \sigma}}^2, |G|_{\mathcal{F}^{\beta + \frac{1}{2}, \sigma}}^2$  and  $\mathbb{E}|F_1(0)|^2$  and  $\mathbb{E}|A^\gamma \xi|^2$ . In addition,  $X$  satisfies the estimate

$$\mathbb{E}|X(t)|^2 + t^{2\gamma} \mathbb{E}|A^\gamma X(t)|^2 \leq C_{F_1, F_2, G, \xi}, \quad t \in [0, T_{F_1, F_2, G, \xi}],$$

with some constant  $C_{F_1, F_2, G, \xi}$  depending only on  $|F_2|_{\mathcal{F}^{\beta, \sigma}}^2, |G|_{\mathcal{F}^{\beta + \frac{1}{2}, \sigma}}^2, \mathbb{E}|F_1(0)|^2$  and  $\mathbb{E}|A^\gamma \xi|^2$ .

Let us finally verify continuous dependence of solutions on initial data.

**THEOREM 5.8.** *Let (4.2), (4.3), (H1'), (H2) and (H3) be satisfied. Let  $X$  and  $\bar{X}$  be the solutions of (5.1) for the data  $(F_2, G, \xi)$  and  $(\bar{F}_2, \bar{G}, \bar{\xi})$  in  $\mathcal{B}_1 \times \mathcal{B}_2 \times B_A$ , respectively, where  $\mathcal{B}_1, \mathcal{B}_2$  and  $B_A$  are defined by (5.37), (5.38) and (5.39). Then*

$$\begin{aligned} & t^{2\eta} \mathbb{E}|A^\eta[X(s) - \bar{X}(s)]|^2 + t^{2\eta} \mathbb{E}|X(s) - \bar{X}(s)|^2 \\ & \leq C_{\mathcal{B}_1, \mathcal{B}_2, B_A} [\mathbb{E}|\xi - \bar{\xi}|^2 + t^{2\beta} |F_2 - \bar{F}_2|_{\mathcal{F}^{\beta, \sigma}}^2 + t^{2\beta} |G - \bar{G}|_{\mathcal{F}^{\beta + \frac{1}{2}, \sigma}}^2], \end{aligned}$$

and

$$\begin{aligned} & t^{2(\eta-\beta)} [\mathbb{E}|A^\eta[X(s) - \bar{X}(s)]|^2 + \mathbb{E}|X(s) - \bar{X}(s)|^2] \\ & \leq C_{\mathcal{B}_1, \mathcal{B}_2, B_A} [\mathbb{E}|A^\beta(\xi - \bar{\xi})|^2 + |F_2 - \bar{F}_2|_{\mathcal{F}^{\beta, \sigma}}^2 + |G - \bar{G}|_{\mathcal{F}^{\beta+\frac{1}{2}, \sigma}}^2] \end{aligned}$$

for  $t \in (0, T_{\mathcal{B}_1, \mathcal{B}_2, B_A}]$ .

As the proof of this theorem is quite analogous to that of Theorem 5.4, we may omit it.

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